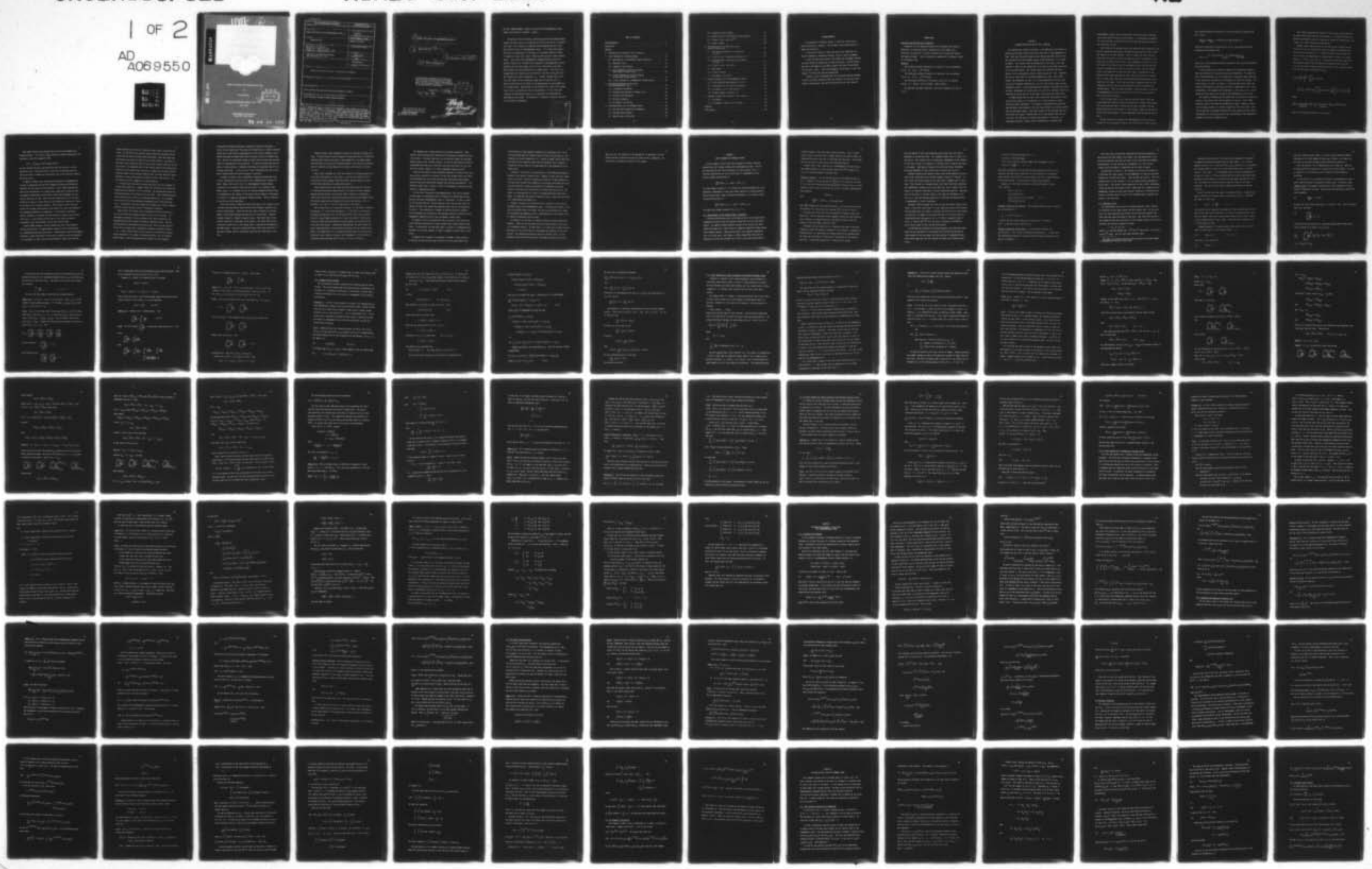
AD-A069 550 NORTH CAROLINA UNIV AT CHAPEL HILL INST OF STATISTICS F/G 12/1
QUEUES AND STORES WITH NONHOMOGENEOUS INPUT.(U)
APR 79 D K MCCLISH N00014-76-C-0550
UNCLASSIFIED MIMEO SER-1225 NL

1 OF 2
AD
A069550



[redacted]

QUEUES AND STORES WITH NONHOMOGENEOUS INPUT

by

Donna McClish

Institute of Statistics Mimeo Series #1225

April 1979

DEPARTMENT OF STATISTICS
Chapel Hill, North Carolina

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Queues and Stores with Nonhomogeneous Input		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
		6. PERFORMING ORG. REPORT NUMBER Mimeo Series No. 1225
7. AUTHOR(s) Donna McClish		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0550
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics, University of North Carolina, Chapel Hill, North Carolina 27514		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Statistics & Probability Program, ONR, Arlington, VA 22217		12. REPORT DATE April 1979
		13. NUMBER OF PAGES 165
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release: Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Ph.D. dissertation under the direction of Walter L. Smith

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Limit theorems, Storage Systems, Queueing Theory

79 06 04 062

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

It is shown that under a variety of circumstances, the presence of a periodic pattern of input is enough to insure that a quasi-limiting distribution of store content exists. For stores with nonhomogeneous compound Poisson input and general release rule a probabilistic approach is developed which allows equations to be written down easily for such quantities as the mean store content and the probability of an empty store. For at least some instances of the M(t)/M/1 queue it is shown that the approximations are very good.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

QUEUES AND STORES WITH NONHOMOGENEOUS INPUT.

Doctoral thesis,

MIMEO SER-1225

by

Donna Katzman McClish

A Dissertation submitted to the faculty of The University of North Carolina at Chapel Hill in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Statistics.

Chapel Hill

174p.

Apr 1979

Approved by:

Adviser

Reader

Reader

This research was partially supported by the Office of Naval Research under Grant N00014-76-C-0550

410 064

4B

McCLISH, DONNA KATZMAN. Queues and Stores With Nonhomogeneous Input. (Under the direction of WALTER L. SMITH.)

Virtually all the literature concerning queueing and storage theory assumes that the input to a system has an arrival rate which is constant over time. This research is concerned with extending results to the more realistic case of nonhomogeneous input. It is shown that under a variety of circumstances, the presence of a periodic pattern of input is enough to insure that a quasi-limiting distribution of store content exists. For stores with nonhomogeneous Compound Poisson input and general release rule a probabilistic approach is developed which allows equations to be written down easily for such quantities as the mean store content and the probability of an empty store. To do this it is assumed that the intensity can be written in the form $\lambda(t)=\lambda_0(1+\varepsilon\phi(t))$. The formulas are series expansions in ε. A more in-depth look at the first two terms of the ε-expansion shows that for periodic $\phi(t)$, a general expression can be found for an approximation to the probability of an empty store. Applications of the formulas for stores with release rules $r(x)\equiv c$, $r(x)=a+bx$ and $r(x)=\alpha x$ are examined. For at least some instances of the M(t)/M/1 queue it is shown that the approximations developed are very good. The problem of a stochastic intensity function is also briefly considered.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I am grateful to Professor Walter L. Smith for serving as my adviser during this research. He provided a never ending source of ideas and directions.

I also wish to thank all those who were on my committee, Professors Leadbetter, Shachtman, Ehrhart, Carroll, Kelly and Wegman.

I have received much support from friends and colleagues through the years which has enabled me to reach this point. Special thanks must go to the people on the SENIC project with whom I have worked for the past 3 years. Their encouragement, particularly in the final stages of writing, was greatly appreciated.

Finally, I must acknowledge the love and understanding extended to me by my family and particularly my husband, Tom, who was always there to reassure me, even while wrestling with his own research.

CONVENTIONS

Numbering and Referencing of Equations

Equations will be numbered consecutively throughout each chapter. References to equations within a chapter will be by number only. References to equations in earlier chapters will include both chapter and equation number. Thus a reference to equation 25 in Chapter 3 would be to equation 3.25.

Notation

The Laplace-Stieltjes Transform of a function F will be denoted by $F^*(s)$; i.e. $F^*(s) = \int_0^\infty e^{-sx} dF(x)$.

The (ordinary) Laplace Transform of a function F will be denoted $F^o(x)$; i.e. $F^o(s) = \int_0^\infty e^{-sx} F(x) dx$.

The Stieltjes Convolution of functions F and G will be denoted $F*G(x)$; i.e. $F*G(x) = \int_0^x F(x-u)dG(u) = \int_0^x G(x-u)dF(u)$.

We note that the above functions F and G are assumed to be zero on $(-\infty, o)$.

CHAPTER 1

INTRODUCTION AND REVIEW OF THE LITERATURE

The problem this research considers is the modeling of and effect on queueing and storage systems which have input that is not homogeneous in time. Most work concerning queueing and storage processes has been for the case where the times between inputs to the system are i.i.d. random variables. This implies that the probability of an input in one time interval is equal to the probability of an input in any other time interval of the same length. But in many practical situations the input process is of a more general nature, often varying with time. Sometimes the arrival rate seems to fluctuate randomly over time. More often, though, the dependence on time is of a more deterministic nature, with the arrival frequency depending on, say, the time of day, month or year. Automobile traffic, airline departures and landings, restaurant and resort business, telephones--all these exhibit a cyclical or periodic pattern of input and are quite typical of the types of processes one often finds.

A common method that has been used to deal with the problem of time varying input (when it is recognized) is to treat periods of widely differing inputs separately. For example, in queueing theory much has been written on the problem of the rush hour-referred to as heavy traffic and non-stationary queues. Such work focuses on the time when the rate of arrival to a queue is almost equal to or even greater than the service rate--not the typical situation the majority of the time. If reasonably practical results could be obtained for a process with

nonhomogeneous input, then we could model the entire system simultaneously, which would certainly be preferable. To understand the complexity of the problem, we should first look at what has been discovered in the literature to date.

The literature of queueing theory has developed quite separately from that of storage and dam theory, although the two areas are often different aspects of the same problem. This can readily be seen if we consider that the input to a store (or dam) in $(0,t]$ is equivalent to the total service time, or work load, of all customers to arrive in $(0,t]$; the content of a store at time t corresponds to the (virtual) waiting time of a customer who arrives at a queue at time t, and the release rate of the content of a store, per unit time, is equivalent to the amount of work load processed per unit time by the server of a queue. Thus, in many cases, what has been discovered about one process can be applied to the other. Still, the work done in queueing theory on nonhomogeneous arrivals has not been picked up by those concerned with storage theory. There has been relatively little published in the area of nonhomogeneous processes and virtually all that has been done has been in the area of queues.

The first major work in this area was done by Takács (1955) who investigated the virtual waiting time process of a single server queue. The virtual waiting time is defined as the time a customer arriving would have to wait to be served. If the system is empty at the time of arrival then the customer is served immediately and the waiting time is zero.

For the single server queue with nonhomogeneous Poisson arrivals (intensity $\lambda(t)$) and general service time distribution, Takács derived

the integro-differential equation for the distribution of waiting time at t, $F(t,x)$:

$$\frac{\partial F(t,x)}{\partial t} = \frac{\partial F(t,x)}{\partial x} - \lambda(t)F(t,x)+\lambda(t)\int_0^x H(x-y)d_yF(t,y).$$

From this he derived an expression for $\Phi(t,s)$, the Laplace-Stieltjes Transform of the waiting time:

$$\Phi(t,s) = e^{st - [1-\psi(s)]\Lambda(t)}\left[1-s\int_0^t e^{-su+[1-\psi(s)]\Lambda(u)} F(u,o)du\right].$$

where $\psi(s)$ is the Laplace-Stieltjes Transform of the service distribution and $\Lambda(t)=\int_0^t\lambda(u)du$.

If the probability the system is empty can be determined uniquely, then the distribution of waiting time can be determined. As Takács admits, the determination of this probability is generally quite difficult. In the particular case where $\lambda(t)$ converges to a positive constant λ for large values of t, and $\lambda\alpha<1$ (where α is the expected service time) the limiting distribution of waiting time exists, independent of the initial distribution. The probability of emptiness in this case is $1-\lambda\alpha$, which agrees with the results for the queue with homogeneous arrival rate λ.

The problem of determining the probability of emptiness has been pursued by a few authors with varying degrees of success. Reich (1958, 1959) was able to show that the probability a queue is empty at time t is the unique solution of a Volterra equation of the first kind. With this he was able to examine the behavior of the probability for large t, culminating in a result which gives the time average of the probability subject to an error of magnitude o(1).

Gani (1962) approached the problem in the context of dam processes. He showed that the probability of emptiness is a function of the first emptiness probability, which in the case of inputs of constant unit size reduces to a simple recurrence relationship.

Hasofer (1964, 1965) was able to use the results of Reich to get explicit results for the M(t)/G/1 queue by assuming a particular form for the Poisson parameter $\lambda(t)$. Taking $\int_0^t \lambda(u)du = t - zb(t)$ Hasofer was able to write the probability of an empty queue at time t as a power series in z, $P(x,t) = \sum_{n=0}^{\infty} Z^n F_n(t)$. Using Reich's equations, some intricate complex analysis and the additional assumption that b(t) and b'(t) are uniformly bounded, Hasofer was able to find a general expression for the Laplace Transform of the F_n:

$$F_n^{\,o}(p) = \frac{1}{G_o^o(p)} \left\{ \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} b_n(p-\sigma) g_n(\sigma) d\sigma \right.$$

$$\left. - \sum_{k=1}^{n} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} b^o_{k-r}(p-s) G_k(s) \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} b^o_r(s-\sigma) F^o_{n-k}(\sigma) d\sigma ds \right\}$$

where

$$G_n^o(p) = \frac{1}{n!} \frac{[1-\psi(\eta)]^n}{[1+\psi'(\eta)]}; \quad g_n^o(p) = \frac{\hat{\Omega}}{n!}(\eta) \frac{[1-\psi(\eta)]^n}{[1+\psi'(\eta)]}; \quad b_n^o(p) = \int_0^{\infty} e^{-pt} [b(t)]^n dt$$

and η is the unique solution of $s - 1 + \psi(s) - p = 0$.

Much neater results were obtained when the Poisson parameter was assumed periodic. For $\lambda(t)=t-z\sum_{n=1}^{n} a_n \sin(n\omega t+\phi_n)$ Hasofer showed that the functions F_n have the asymptotic form

$$F_n(t) \sim \sum_{k=1}^{\infty} \left[A_{kn} \cos(k\omega t)+B_{kn}\sin(k\omega t)\right]$$

and the Laplace-Stieltjes transform of the waiting time has a similar asymptotic form. The coefficients in the Fourier expansion were not given explicitly, although for the special case $\lambda(t)=1-z\omega\cos(\omega t)$ Hasofer derived F_0 and F_1.

A number of authors have approached the problem of nonhomogeneous arrivals by employing the familiar Kolmogorov difference-differential equations for the queue size distribution. Clark (1956) explored the queue size distribution of a single server queue with Poisson arrivals and negative exponential service times, where both parameters are continuous functions of time. While no exact representation for the distribution of queue size was obtained, the problem was reduced to finding the solution of an integral equation. Much simpler results were presented for the mean and variance of the queue size, which only depend on the (unknown) probability of emptiness. In the special case where traffic intensity is constant the distribution of queue size was found explicitly. It can readily be seen that in this case the queue is equivalent to one with constant arrival and service rates.

Luchak (1956) treated a similar problem, but with a more general service distribution. He approximated a general service distribution by a Weighted-sum Erlang distribution. Starting again from the Kolmogorov equations, Luchak found that if the traffic intensity could be expressed as a polynomial in time, then the distribution of queue size could be

found recursively, but only in selective cases could a closed form be found. At the end of his article, Luchak noted that periodic arrival rates could be dealt with more easily than most, since the queue size distribution need only be found for the initial period. The state at the end of each period can then be used as the initial condition for the next period. He suggested that if the initial state of the system were taken to be the steady state solution of a queue with constant traffic intensity equal to the average intensity over a period, then the "quasi-stationary" steady state would be reached in only a few periods. We will make use of this suggestion in later chapters.

In recent years the work of Luchak and Clarke has been expanded to include more generality. Koopman (1972) was interested in arrivals and departures of airplanes at an air terminal. He incorporated certain of the peculiarities of the airport into his model. These included periodicity (24 hours) and a limit on the maximum queue size. He also made arrival and service time parameters dependent both on time and queue size. The method of solution suggested by Koopman was a direct computer solution of the (finite) system of differential equations.

Moore (1975) like Luchak, used a Weighted-sum Erlang distribution for service times and assumed customers could arrive in groups. He developed a technique, in the stationary case, which focused on the embedded Markov chain formed by looking at departure times. Then by approximating the continuously varying arrival rate by a step function he was able to extend this, developing a computer algorithm solution.

E. L. Leese (1967) wrote a paper in which he evaluated the computational feasability of a number of proposed solutions for the nonhomogeneous queue. Leese considered direct solution of the Kolmogorov

differential-difference equations, generating function techniques, a Taylor series expansion of the queue size probabilities, a matrix approach which uses a step function approximation to the traffic intensity, and a method developed by Wragg (1963) which involves solving an integral equation. Most of the objections raised by Leese revolve around the excessive number of computational steps required. It should be noted that over ten years of rapid advances in computer technology has abrogated some of his original objections. In fact, we will later investigate the uses of the matrix approach when the arrival pattern is periodic.

A number of authors have investigated a queueing problem where the arrival (and possibly service) rate is not a deterministic function of time. Rather, the arrival rate is a heterogeneous Poisson process governed by an "extraneous phase process" which is a continuous time Markov chain, i.e. customers arrive in a Poisson stream with the arrival rate depending on the "phase" of the queue, and the amount of time spent in a phase is a negative exponential random variable. Service times may or may not vary with each phase.

Eisen and Tainiter (1963) and Yechiali and Naor (1971) looked at such a process with two phases, where service also followed a negative exponential distribution. Eisen and Tainiter found a formula for the mean number in the system and mean waiting time in steady state. Yechiali and Naor looked at a number of special cases. They found that if the traffic intensity in each phase of the queue is the same, then the distribution of queue size in the steady state is what would be expected in the M/M/1 queue. They also claimed the above result would hold only in the case of equal traffic intensities, which we will show later is not true.

Yechiali (1973) later extended his result to include an m-phase process. To find explicit results requires finding the roots of a difficult equation which yields that nemesis, the probability of an empty queue. No closed form solutions were presented except in the special case where traffic intensity of each phase are all equal, with results as before for $m=2$.

Neuts (1971) extended this work even further by allowing general service time distributions in each phase. By using very powerful tools along with a Markov renewal branching process approach Neuts was able to find both transient and steady state results.

Three separate papers have been published concerning the existence of limiting distributions when arrivals to a queue are from a nonhomogeneous Poisson process--one by Sakr (1971), another by Yevdokimova (1974) and a third just recently published by Harrison and Lemoine (1977). All three assume that the intensity of the arrival process, $\lambda(t)$, is periodic, i.e. $\lambda(t+\tau) = \lambda(t)$ for some period τ and every t.

Sakr proved that for a queue with m servers, if the traffic intensity per period, λ/μ is less than one (where $\lambda = \frac{1}{\tau}\int_0^\tau \lambda(u)du$ is the average intensity per period and $1/\mu$ is the mean service time) then the distribution of queue length converges to a "quasi-stationary" limiting distribution, where the distribution of queue length is a function of the time within the period. Yevdokimova expanded upon the work of Sakr to give a representation of the waiting distribution for the single server queue. He showed that if $\lambda/\mu<1$ then the waiting time is the sum of two functions, one of which is periodic and independent of the initial distribution, while the other goes to zero as t goes to infinity.

The methods used in both articles are virtually identical. Both authors used two queues with related arrival processes to bound the original process. They were then able to use classical theory for the homogeneous queue to get limiting results. To get his more explicit results, Yevdokimova also made use of the fact that the time points $\{n\tau\}_{n=1}^{\infty}$ such that the system is empty at $n\tau$ form an embedded renewal process.

Harrison and Lemoine, with assumptions identical to those of the first two articles, proved the existence of both time average limits and limits in distribution for the virtual waiting time and waiting time process. They also derived relationships between the limiting distributions. The method of proof used is similar to that of Yevdokimova, focusing on empty times as a regenerative event.

As this brief literature review demonstrates, very little progress has been made in developing practical formulae for determining virtual waiting time when nonhomogeneous input is considered. In fact, no work at all has been done for storage processes with time varying input and general release rules beyond the queueing theory results corresponding to $r(x)=c$. This research will attempt to develop practical formulae for the probability of emptiness and expected content of stores with nonhomogeneous Compound Poisson input and general release rate.

In Chapter 2 we are concerned with the existence of the limiting or quasi-limiting distribution of store content under rather general conditions. In particular, we show that under a variety of circumstances the presence of a periodic pattern of input is enough to ensure that a limit exists.

A probabilistic approach is developed in Chapter 3 which allows us to quickly and easily write down equations for such quantities as the

distribution of store content, probability of emptiness, etc. To do this we assume that the intensity has the form $\lambda(t)=\lambda_o(1+\varepsilon\phi(t))$. The formulae are series expansions in ε. A more in depth look at the first two terms of the expansion shows that for periodic $\phi(t)$, a general expression can be found for an approximation to the probability of an empty store.

Chapters 4 and 5 focus on applications of the formulae developed in Chapter 3 for specific release rules. Chapter 4 treats the most common rule, $r(x)\equiv 1, x>0$. Specific formulae are given for an approximation to the mean store content and probability of an empty store when the input size distribution is negative exponential or Weighted-sum Erlang. Chapter 5 considers the release rules $r(x)=a+bx$ and $r(x)=cx, x>0$. The latter is representative of a class of storage processes which never empty. We show that the expected content of such a store can be written as a second degree polynomial in ε.

Chapter 6 presents an alternative matrix approach to the study of the M/M/1 queue with nonhomogeneous input. Its particular appeal in the case of periodic input is pursued. We also use this method as a means to investigate the adequacy of the ε-approximations in the special case of negative exponential service time.

In Chapter 7 we consider a slightly different problem from earlier chapters. There we are concerned with the situation where $\lambda(t)=\lambda(t,\omega)$ is a stochastic process. We show that it is sufficient to know the mean function of $\lambda(t,\omega)$ to be able to investigate the behavior of the store content. The problem presented by Yechiali and Naor (1971) is also examined, with the surprising result that when the mean service time

does not vary, the formula for the probability of emptiness is quite simple and only involves the first two terms of the ε-expansion. We also correct a mistake by Yechiali in his papers.

CHAPTER 2

LIMIT THEOREMS FOR STORAGE SYSTEMS

In this chapter we will prove the existence of certain limiting probabilities for storage systems with nonhomogeneous input. We will be concerned here with the distribution of store content, $Z(t)$. A limiting distribution for $Z(t)$ exists which is independent of the initial content distribution if

$$\lim_{t\to\infty} P\{Z(t) \leq x \mid Z(0)\} = P\{Z^* \leq x\}$$

for some random variable Z^*. If instead the limiting probability is not completely independent of time, but rather exhibits a cyclical pattern, then we say we have a quasi-limiting distribution, i.e. we have a quasi-limiting distribution if

$$\lim_{k\to\infty} P\{Z(k\tilde{\omega}+\tau) \leq x) \mid Z(0)\} = P\{Z^*(\tau) \leq x\}$$

for some $\tilde{\omega}$ and random variables $Z^*(\tau)$, $0 \leq \tau < \tilde{\omega}$.

2.1 Some Results in the Storage Theory Literature

Limiting distributions were first shown to exist for the queue with homogeneous Poisson input and negative exponential service time distribution. The general method employed focused on the states $\{E_j\}$ which denote the queue size. When service is negative exponential these states form a Markov process. When service time is not negative exponential this is not the case. But Takács, in 1955, showed that this was not necessary to prove the existence of a limit, since the waiting time is

a Markov process if arrivals from a Poisson process. Even if inter-arrival times are merely from a renewal process, the arrival times are regeneration points, and this is sufficient for the proof. Thus Takács was able to prove results such as the following theorem.

Suppose $\lambda(t)$ is the intensity of a nonhomogeneous Poisson process, the random variable χ_n is the service time of the n^{th} customer, and $F(x,t)$ is the distribution of waiting time.

<u>Theorem 3 (Takács)</u>: Let the expected value of the random variable χ_n be $\alpha = \int_0^\infty x dH(x) = \int_0^\infty (1-H(x))dx$ and let $\lim_{t\to\infty} \lambda(t)=\lambda$ be a positive constant. If $\lambda\alpha<1$ then the limiting distribution $\lim_{t\to\infty} F(t,x)=F^*(x)$ exists, is independent of the initial distribution $F_0(x)$ and is uniquely determined by the equations

$$F^*(0)=1-\lambda\alpha$$

and

$$\frac{dF^*(x)}{dx} = \lambda\left[F^*(x) - \int_0^\infty H(x-y)dF^*(y)\right]$$

where $\frac{dF^*}{dx}$ denotes the right hand derivative. If $\lambda\alpha>1$ then the limiting distribution $F^*(x)$ does not exist, however $\lim_{t\to\infty} F(t,x)=0$ for each x.

Of course this result can also be applied to standard storage and dam theory. No new limiting results appeared concerning these processes until interest in storage processes with general release rules was aroused in the late 1960's and early 1970's.

A release rule (or rate) $r(u)$ is a function such that in any time interval $(t,t+dt)$ the amount of content in a store which is released is $r(Z(t))dt+o(t)$; i.e. the amount of content that will be released from the store per unit time is a function of the content of the store at that time. In queueing theory this is equivalent to saying

that the amount of work load processed by the server per unit time is dependent on the waiting time. The standard release rule is $r(u) \equiv c$, $u>0$ and $r(0)=0$. This release rule is essentially independent of the content, as the store content is released at a constant rate as long as the store is not empty. Some general release rules which have been considered which are dependent on store content are $r(x)=a+bx$ and $r(x)=cx^{\alpha}$.

Storage processes with general release rules which depend on the content of the system have more recently become a subject of interest. In 1971 Cinlar and Pinsky opened up the field with a landmark work in this area. They focused on a storage process with input which has stationary and independent increments and finite jump rate, and a release rule $r(x)$ which is a Lipschitz continuous, strictly increasing function. They were able to show that if the mean input is less than the supremum of $r(x)$ over all x, then the storage content has a limiting distribution which is independent of initial conditions.

Brockwell (1977) extended this work to show that under certain conditions, a store with input which is a pure jump Levy process (with perhaps infinite jump rate) will admit a stationary distribution of store content. In particular, in Lemma 5.4 he proved that for $r(x)$ strictly positive, nondecreasing and continuous on $(0,\infty)$, if $r(0+)>0$, expected input in $(0,t)$ A_t finite, and $\sup_x r(x) > EA_1$ then a stationary distribution exists and is equal to the limiting distribution.

In 1976 Harrison and Resnick derived necessary and sufficient conditions for the existence of a stationary and limiting distribution of store content in the case where the input process is Compound Poisson with finite jump rate, and the release rule meets the following conditions:

i) $r(x)$ is strictly positive for $x > 0$

ii) $r(x)$ is left continuous

iii) r has a strictly positive right limit everywhere in $(0,\infty)$

iv) $R(x)=\int_0^t \frac{dy}{r(y)} < \infty \quad 0 < x < \infty$

With these assumptions not only were Harrison and Resnick able to find necessary and sufficient conditions for the existence of a stationary (and limiting) distribution of store content, but also a representation of the distribution. They proved the following two theorems:

Suppose λ is the input rate and $F(\ )$ the distribution of input size. Define

$$Q(x)=\lambda\,(1-F(x))$$
$$K(x,y)=Q(x-y)/r(x)$$
$$K_{n+1}(x,y)=\int_y^x K(x,z)\,K_n(z,y)dz \qquad 0 \leq y < \infty$$
$$K^*(x,y)=\sum_{n=1}^{\infty} K_n(x,y)$$

<u>Theorem 1 (Harrison and Resnick)</u>: The contents process X_t has a stationary distribution iff

$$\frac{1}{\gamma_0} = 1 + \int_0^\infty K^*\,(x,o)\;dx < \infty$$

in which case the unique stationary distribution γ has density $g(x) = \gamma_0\, K^*(x,0)$ on $(0,\infty)$ and $\gamma(0)=\gamma_0$.

<u>Theorem 2 (Harrison and Resnick)</u>: X_t is positive recurrent iff $\int_0^\infty K^*(x,0)dx < \infty$ iff X_t has a stationary distribution γ. In this case the limiting distribution Π coincides with the unique stationary distribution in Theorem 1.

While the input distribution considered by Harrison and Resnick is more restrictive than before, the release rule considered here is certainly more general than that of either Cinlar and Pinsky or Brockwell. Unfortunately, as they pointed out, assumption (iv) rules out any release rule which makes it impossible for the store to empty.

As mentioned in Chapter 1, the problem of proving the existence of a limit when the input process is not homogeneous in time has only been examined for the M(t)/G/1* queue. There we have two results. The first, stated earlier in this chapter, is for an intensity which approaches a finite limit. The second, proved separately by Sakr (1972), Yevdokimova (1974) and Harrison and Lemoine (1977), shows that for a queue with non-homogeneous Poisson arrivals with periodic intensity, a limiting distribution of waiting time will exist if the average traffic intensity per period is less than one.

2.2 Some Basic Tools

We extend these results here to storage processes under a variety of conditions on the input process and the release rule. For the most part, the input processes we consider will be much more general than those that have been considered in the past. One restriction we add is the assumption that each realization of the storage process can be represented by the infinite sequence of inputs

$$\omega = \{X_1, U_1, X_2, U_2, \ldots\}$$

where X_i is the time between the $i\text{-}1^{th}$ and i^{th} input and U_i is the size of the i^{th} input. We call such input discrete input.

---

*We adapt the standard queueing theory notation to let M(t) denote a nonhomogeneous Poisson arrival rate.

A method of proof that will be used often throughout this chapter is to compare two storage processes with the same input ω, i.e., we will compare $Z_1(t)=Z(t,\omega,\xi,r,)$ and $Z_2(t)=Z(t,\omega,\xi_2,r_2)$ where $Z(t,\omega,\xi,r)$ denotes the storage content at time t of a store with release rule r, initial content ξ, and input ω. In forthcoming proofs various assumptions will be made on ξ_i and r_i. But by assuming that for a given realization both processes receive inputs at precisely the same moment and have the same size, we remove any differences which could be caused by the input structure. Thus we can more readily compare the contents of the two processes.

We will consider here two types of storage processes. A Type A process is defined as a storage process with release rule $r(x)$ such that for some "a", $r(a)=0$ and

(1) $$R(a,x) = \int_a^x \frac{du}{r(u)} < \infty \qquad 0 < x < \infty$$

The function $R(a,x)$ represents the amount of time it would take the content of the store to go from level X to level a if there were no input. Thus (1) implies that for a Type A process there exists a level a which can be reached from any level X above a (with positive probability) in a finite amount of time.

A Type B process is a storage process with release rule $r(x)$ such that for some level a, $r(a)=0$ and for every $\varepsilon > 0$

(2) $$R(a+\varepsilon,x) < \infty \qquad a + \varepsilon < x < \infty$$

but there is an X such that

(3) $$R(a,x) = \infty$$

Thus for a Type B process there is a level a which cannot be reached from above in a finite amount of time, even if there is no input, but the level of the store content can get arbitrarily close to a.

Note that without loss of generality we can assume $a=0$. When $a>0$ we simply have the situation of a store with a "false bottom". For sake of simplicity, in the future we will assume $a=0$ unless otherwise specified.

At this point it is convenient to derive a few properties of Type A and B storage processes with general release rules. We define a <u>consumption curve</u> as the graph a realization of store content $Z(t)$ would follow if there were no input. These curves are described by the differential equation

$$\frac{dZ(t)}{dt} = -r\,(Z(t)) \tag{4}$$

We take $Z(t)$ to be right continuous, i.e. $Z(t+0) = Z(t)$. Thus (4) implies that $Z(t-)$ satisfies

$$\int_{Z(t-)}^{Z(t_o)} \frac{du}{r(u)} = t - t_o \tag{5}$$

From (5) we can see that $Z(t)$ is continuous except where inputs occur, since if there is no input in $(t_o - \frac{1}{n}, t_o)$

$$\lim_{n\to\infty} \int_{Z(t_o)}^{Z(t_o - \frac{1}{n})} \frac{du}{r(u)} = \lim_{n\to\infty} \left[t_o - (t_o - \tfrac{1}{n})\right] = 0$$

i.e. $Z(t_o-0) = Z(t_o)$.

Thus we have that the consumption curves are continuous and strictly decreasing functions of t passing through the point $(t_o, Z(t_o))$ and every positive level less than $Z(t_o)$. The latter is true since for a Type A or B process

$$\int_a^b \frac{dx}{r(x)} < \infty \qquad 0 < a < b$$

We prove now some Lemmas concerning the storage processes.

Lemma 2.1.1: Let $Z_1(t) = Z(t,\omega, \xi_1,r)$ and $Z_2(t) = Z(t,\omega, \xi_2,r)$ be the content of two storage processes. If for some t_o $Z_2(t_o) \geq Z_1(t_o)$ then $Z_2(t) \geq Z_1(t)$ for all $t > t_o$

Proof: Let t_1 be the time of the first input after t_o. We first show that $Z_2(t) \geq Z_1(t)$ for $t_o \leq t < t_1$. In the case $Z_1(t_o) \leq Z_2(t_1)$ this is clear, since $Z_1(t_1) \leq Z_1(t_o) \leq Z_2(t_1)$. Thus we need to look at the case $Z_2(t_1) < Z_1(t_o)$. Suppose the conclusion we seek does not hold and $Z_2(t) < Z_1(t)$ $t_o \leq t < t_1$. Then

$$t-t_o = \int_{Z_2(t)}^{Z_2(t_o)} \frac{du}{r(u)} = \int_{Z_1(t_o)}^{Z_2(t_o)} \frac{du}{r(u)} + \int_{Z_1(t)}^{Z_1(t_o)} \frac{du}{r(u)} + \int_{Z_2(t)}^{Z_1(t)} \frac{du}{r(u)}$$

But we also have

$$\int_{Z_1(t)}^{Z_1(t_o)} \frac{du}{r(u)} = t - t_o$$

which implies that

$$\int_{Z_1(t_o)}^{Z_2(t_o)} \frac{du}{r(u)} + \int_{Z_2(t)}^{Z_1(t)} \frac{du}{r(u)} = 0.$$

This is impossible since $r(u)>0$ and $Z_2(t_o)>Z_1(t_o)$ and $Z_1(t)>Z_2(t)$. Thus we can conclude that $Z_2(t) \geq Z_1(t)$ for $t_o \leq t<t_1$.

Suppose at t_1 there is an input of size U. We know

$$Z_2(t_1-) \geq Z_1(t_1-)$$

Then

$$Z_2(t_1) = Z_2(t_1-) + U \geq Z_1(t_1-) + U = Z_1(t_1)$$

Thus we know that $Z_2(t) \geq Z_1(t)$ both between inputs and directly after inputs, which is at all times; i.e. we can conclude

$$Z_2(t) \geq Z_1(t) \qquad \text{for all } t$$

<u>Lemma 2.1.2</u>: Suppose r(u) is nondecreasing. Then

$$\int_{a+c}^{b+c} \frac{du}{r(u)} \leq \int_a^b \frac{du}{r(u)} \qquad a,b,c>0$$

<u>Proof</u>: For the integral $\int_{a+c}^{b+c} \frac{du}{r(u)}$ we make the substitution $y=u-c$. This gives

$$\int_{a+c}^{b+c} \frac{du}{r(u)} = \int_a^b \frac{dy}{r(y+c)}$$

then

$$\int_{a+c}^{b+c} \frac{du}{r(u)} - \int_a^b \frac{du}{r(u)} = \int_a^b \frac{dy}{r(y+c)} - \int_a^b \frac{dy}{r(y)}$$

$$= \int_a^b \frac{r(y)-r(y+c)}{r(y)\ r(y+c)}\ dy$$

Since $r(y)$ is nondecreasing $r(y) \leq r(y+c)$. Thus we have

$$\int_{a+c}^{b+c} \frac{du}{r(u)} \leq \int_{a}^{b} \frac{du}{r(u)}$$

<u>Lemma 2.1.3</u>: Let $Z_1(t) = Z(t,\omega,\xi_1,r_1)$ and $Z_2(t) = Z(t,\omega,\xi,r_2)$ be the contents of two Type A or B storage processes, with $r_1(x) \leq r_2(x)$. If for some t_o, $Z_1(t_o) \geq Z_2(t_o)$ then $Z_1(t) \geq Z_2(t)$ for all $t \geq t_o$.

<u>Proof</u>: Let t_1 be the time of the first input after t_o. For $t_o<t<t_1$

$$t - t_o \qquad \int_{Z_1(t)}^{Z_1(t_o)} \frac{du}{r_1(u)} \geq \int_{Z_1(t)}^{Z_1(t_o)} \frac{du}{r_2(u)}$$

since $r_1(x) \leq r_2(x)$. The assumption that $Z_1(t_o) \geq Z_2(t_o)$ implies that

$$\int_{Z_1(t)}^{Z_1(t_o)} \frac{du}{r_2(u)} \geq \int_{Z_1(t)}^{Z_2(t_o)} \frac{du}{r_2(u)}$$

Suppose that $Z_1(t)<Z_2(t)$. Then

$$\int_{Z_1(t)}^{Z_2(t_o)} \frac{du}{r(u)} > \int_{Z_2(t)}^{Z_2(t_o)} \frac{du}{r(u)} = t-t_o$$

a contradiction. Thus for $t_o<t<t_1$, $Z_1(t) \geq Z_2(t)$.

Suppose at t_1 there is an input of size U. Then

$$Z_1(t_1) = U + Z_1(t_1-) \geq Z_2(t_1-) + U = Z_2(t_1).$$

Since we have $Z_1(t) \geq Z_2(t)$ in between times of inputs and directly after an input, it is clear that $Z_1(t) \geq Z_2(t)$ for all $t \geq t_0$.

2.3 A General Limit Theorem

We now proceed to results concerning the limiting value of store content. The first theorem shows that under certain circumstances the existence of a stationary distribution is sufficient to insure that a limiting distribution will exist which is independent of the initial distribution.

Theorem 2.1: Let $Z(t)$ be the content of a Type A or B storage process with nondecreasing release rule $r(x)$ and discrete input. Suppose further that for any $\varepsilon > 0$ and initial content ξ there exists a proper random variable $T(\xi,\varepsilon) = T$ such that $Z(T) = \varepsilon$. (If $\xi \leq \varepsilon$ then define $T=0$). If the process has a stationary distribution then it has a limiting distribution which is independent of the initial distribution, and the two distributions are equal.

Proof: Suppose we have two storage processes of Type A or B, $Z_1(t) = Z(t,\omega,\xi_1,r_1)$ and $Z_2(t) = Z(t,\omega,\xi_2,r_2)$ where $r_1(x) = r_2(x)$ is nondecreasing and both processes satisfy the assumptions of the theorem. Let $\xi_1 \geq \xi_2$. By Lemma 2.1.1

$$(6) \qquad Z_1(t) \geq Z_2(t) \qquad \text{for all } t$$

It follows that $T(\xi_1,\varepsilon) \geq T(\xi_2,\varepsilon)$, which together with (6) implies that

$$(7) \qquad 0 \leq Z_1\,(T(\xi_1,\varepsilon)) - Z_2(T(\xi_1,\varepsilon)) \leq \varepsilon$$

Suppose that the first input after $T(\xi_1,\varepsilon)$ occurs at t_1. If we can show (7) holds for all $T(\xi_1,\varepsilon) \leq t \leq t_1$ then clearly it will hold for all $t \geq T(\xi_1,\varepsilon)$.

Let $T(\xi_1,\varepsilon)<t<t_1$. Since there have been no inputs in this interval we still have

(8) $$\varepsilon = Z_i\ (T(\xi_1,\varepsilon)) > Z_i(t) \qquad i=1,2$$

so that

$$0 \leq Z_1(t) - Z_2(t) \leq \varepsilon \qquad \text{for } T(\xi_1,\varepsilon)<t<t_1$$

Now suppose at t_1 we have an input of size U. Then

$$Z_i(t_1)=Z_i(t_1-)+U \qquad i=1,2$$

and we have from (6) and (8) that

$$U \leq Z_2(t_1) \leq Z_1(t_1) \leq U + \varepsilon$$

Hence we can conclude that for all $t > T(\xi_1,\varepsilon)$

$$0 \leq Z_1(t) - Z_2(t) \leq \varepsilon$$

Similarly, if $\xi_1 < \xi_2$ we have that for $t > T(\xi_2,\varepsilon)$

$$0 \leq Z_2(t) - Z_1(t) \leq \varepsilon$$

From which we can conclude that

$$|Z_1(t) - Z_2(t)| \leq \varepsilon \qquad \text{for } t \geq \max\left[T(\xi_1,\varepsilon),\ T(\xi_2,\varepsilon)\right] = T^*$$

We can now easily show that $Z_1(t) - Z_2(t) \to 0$ in probability for

$P\{|Z_1(t)-Z_2(t)| > \varepsilon|\xi_1,\xi_2\} =$

$P\{|Z_1(t)-Z_2(t)| > \varepsilon \text{ and } t \geq T^*|\xi_1,\xi_2\} +$

$P\{|Z_1(t)-Z_2(t)| > \varepsilon \text{ and } t < T^*|\xi_1,\xi_2\} \leq$

$0 + P\{t<T^*\}$

But $T(\xi_i,\varepsilon)$ is proper for each i, and thus so is T^*, which means

$$\lim_{t\to\infty} P\{|Z_1(t)-Z_2(t)| > \varepsilon \mid \xi_1,\xi_2\} = 0$$

Let $F_{\xi_i}(x,t) = P\{Z_i(t) \leq x \mid Z_i(o) = \xi_i\}$ $\qquad i=1,2.$

Since $Z_1(0)$ is independent of $Z_2(0)$ we have

$F_{\xi_1}(x,t)=P\{Z_1(t) \leq x \mid \xi_1,\xi_2\}$

$= P\{Z_1(t) \leq x \text{ and } \mid Z_1(t)-Z_2(t) \mid \leq \varepsilon|\xi_1,\xi_2\}$

$+ P\{Z_1(t) \leq x \text{ and } \mid Z_1(t)-Z_2(t) \mid > \varepsilon \mid\xi_1,\xi_2\}$

$\leq P\{Z_2(t) \leq x + \varepsilon \mid\xi_1,\xi_2| + P\{|Z_1(t)-Z_2(t)| > \varepsilon|\xi_1,\xi_2\}$

i.e.

$$(9) \quad F_{\xi_1}(x,t) \leq F_{\xi_2}(x+\varepsilon,t) + P\{|Z_1(t)-Z_2(t)| > \varepsilon \mid \xi_1,\xi_2\}$$

Suppose now that ξ_i has distribution G_i. From (9) we have, taking expectations

$$F_1(x,t) \leq F_2(x+\varepsilon,t) + E\left[P\{|Z_1(t)-Z_2(t)| > \varepsilon \mid \xi_1,\xi_2\}\right]$$

where $F_i(x,t) = E\, F_{\xi_i}(x,t)$ $\qquad i=1,2.$

We note that by dominated convergence

$$\overline{\lim_{t\to\infty}}\ E\left[\ P\{|Z_1(t)-Z_2(t)\ |\ >\varepsilon\ |\ \xi_1,\xi_2\ \}\ \right]=0$$

Thus

$$(10)\quad \overline{\lim_{t\to\infty}}\ F_1(x,t)\le \overline{\lim_{t\to\infty}}\ F_2(x+\varepsilon,t)$$

Similarly, by interchanging the roles of Z_1 and Z_2 and substituting x-ε for x we can get

$$\underline{\lim_{t\to\infty}}\ F_2(x-\varepsilon,t)\le \underline{\lim_{t\to\infty}}\ F_1(x,t)$$

By assumption a stationary distribution H exists for each storage process. Thus we may let $G_2(x) = H(x)$. Then $H(x) = F_2(x,t)$ for all t, and from (10)

$$\overline{\lim_{t\to\infty}}\ F_1(x,t)\le H(x+\varepsilon)$$

Letting ε go to zero we can get

$$\overline{\lim_{t\to\infty}}\ F_1(x,t)\le H(x+0)$$

Similarly

$$H(x-0)\le \underline{\lim_{t\to\infty}}\ F_1(x,t)$$

Thus

$$H(x-0)\le \underline{\lim_{t\to\infty}}\ F_1(x,t)\le \overline{\lim_{t}}\ F_1(x,t)\le H(x+0)$$

At each continuity point of H we have

$$\lim_{t\to\infty} F_1(x,t)=H(x)$$

the desired conclusion.

2.4 Limit Theorems for Type A Processes with General Periodic Input

Theorem 2.1 requires a very strong assumption, the existence of a stationary distribution, which in itself is often difficult to verify. The next theorem, and those which follow, will not require such a strong assumption. In exchange, more structure will be put on the input process.

We suppose there is a number $\tilde{\omega}$ (called the period) and in each interval $[k\tilde{\omega}, (k+1)\tilde{\omega})$ there are a finite number of times where there is a positive probability of an input to the system. Let these points be denoted.

$$k\tilde{\omega}+\Theta_1 < k\tilde{\omega}+\Theta_2 < \ldots < k\tilde{\omega}+\Theta_p$$

where the Θ_i are the same for each interval. Each point has associated with it a distribution of input size $G_j(x)$, where we allow $G_j(0+)>0$. The Laplace-Stieltjes Transform of these inputs in $(0,t]$ $t=k\tilde{\omega}+\tau$, $0\leq\tau<\tilde{\omega}$, is

$$M_1(s,t) = \left[\prod_{j=1}^{p} G_j^*(s)\right]^k \prod_{j=1}^{l} G_j^*(s)$$

where

$$\Theta_l \leq \tau < \Theta_{l+1}$$

and

$$\prod_{j=1}^{l} G_j^*(s) \text{ is defined to be 1 if } \tau < \Theta_1.$$

We also suppose that in any interval $(t,t + dt)$ there is a probability $\lambda(t)\,dt + o(dt)$ that an input will occur, where $\lambda(t)$ is taken to be a periodic function, i.e. $\lambda(k\tilde{\omega}+\tau) = \lambda(\tau)$, $0\leq\tau<\tilde{\omega}$. The size distribution of these inputs $B(x,t)$ is also taken to be periodic. The Laplace-Stieltjes

Transform of these inputs into the system in $(0,t)$ is

$$M_2(s,t) = \exp\{-\int_0^t (1-B^*(s,u))\,\lambda(u)du\}$$

The total input to the storage system is a convolution of these two inputs, and has Laplace-Stieltjes Transform

$$M(s,t) = M_1(s,t)\,M_2(s,t)$$

We will call any input process which is characterized by this Laplace-Stieltjes Transform, general periodic input or g.p.i. for short. The i^{th} moment of input size will be denoted by β_i.

The g.p.i. is really much more general than we have encountered in the literature. Some storage theory assumes discrete-time input, some have continuous time input, but none have both in combination as we have here.

Another important feature of the g.p.i. is that it is not homogeneous in time. In particular, both the input size distribution and the rate of occurrence of inputs is periodic. Most work in this area has required that inputs be independent of time, with stationary increments. This implies that the distribution of input in one time interval is equal to the distribution of input in any other time interval of the same length, as mentioned earlier. With periodic input, we have that the distribution of input in one time interval which begins at some point τ from the start of a period, $0 \leq \tau < \tilde{\omega}$ is the same for all periods. This fact allows regenerative events to be of use here, as they have been of use in the past for the homogeneous case.

It is clear that a change of time scale can transform the g.p.i. to have period $\tilde{\omega} = 1$. Thus, without loss of generality we will assume throughout the remainder of this work that $\tilde{\omega} = 1$.

<u>Theorem 2.2</u>: Let $Z(t)$ be a Type A storage system with general periodic input and nondecreasing release rule $r(x)$. Define

$$Y(U) = \int_0^u \frac{dx}{r(x)}$$

If

$$\mu_y = \sum_{l=1}^{p} \int_0^\infty Y(u)G_l(du) + \int_0^1 \int_0^\infty Y(x)B(dx,t)\lambda(t)dt < 1$$

then the store content has a quasi-limiting distribution which is independent of the initial distribution.

<u>Proof</u>: Let $Z(t)$ be the store content at time t and define the event ε_τ $0 \le \tau < 1$ as follows: ε_τ occurs at $t=n+1$ (n an integer) if $Z(n+\tau) = 0$. Clearly, ε_τ is a regenerative event, as defined in Smith (1955). Note that ε_τ is periodic with period 1. For a quasi-limiting distribution to exist for $Z(n+\xi)$, $0 \le \xi < 1$ we must show, for a useful class of sets A, that

A1) ε_τ is certain, i.e. ε_τ will occur a first time with probability one.

A2) $\sum_{n=0}^{\infty} \phi(n+\xi)\ (1-F(n+\xi)) < \infty$

where $\phi_A(t-m) = P\{Z(t)\varepsilon A \mid Z(0), n_t>0,\ T_{n_t} = m\}$

n_t = number of occurrences of ε_τ in $[0,t]$

T_{n_t} = time of last occurrence of ε_τ before t

To prove A1 and A2 we will use a series of Lemmas. Before looking at the Lemmas, though, we need to introduce the concepts of empty time and busy time. We define the empty time of the store in the interval $[0,t)$, denoted $e(t)$, as the Lebesgue measure of the set $\{s < t: Z(s) = 0\}$.

In a corresponding manner we define the busy time in the interval $[0,t)$, denoted $b(t)$, as the Lebesgue measure of the set $\{s < t: Z(s) > 0\}$. Notice that $e(t) + b(t) = t$. In addition, we can define a busy cycle. A busy cycle is defined to begin when the store empties, and ends when the store next empties. The initial busy cycle begins at $t=0$ and ends when the store first empties. We now proceed to the Lemmas.

Lemma 2.2.1: Suppose $Z(t)$ is the content of a Type A storage process as described in Theorem 2.2. Then

$$\lim_{n\to\infty} \frac{E[e(n)]}{n} > 0$$

Proof: To prove this lemma we need to construct an alternative storage model which we shall call a warehouse model: The warehouse system will receive the same input ω as the original system. If an input arrives while the alternative store is not empty then the input is stored in a separate warehouse. When the alternative store is about to empty the next input waiting in the warehouse is immediately put into the store to prevent emptiness unless an input arrives at that moment, in which case this input is added to the store, preventing emptiness. Let $Z^A(t)$ be the store content in the alternative warehouse model, and $Z^W(t)$ be the warehouse content.

We claim that the total content in the warehouse system (both store and warehouse) is at least as great as the content in the original model, i.e. $Z^A(t) + Z^W(t) \geq Z(t)$. We prove this by induction on the number of inputs. Suppose inputs occur at times t_1, t_2,... with size $U(t_1)$, $U(t_2)$, ... For $0 \leq t < t_1$ $Z^W(t)=0$ and $Z^A(t)=Z(t)$.

Define $T_o = \inf \{t: Z^A(t) = 0\}$.

Suppose $t_1 < T_o$. At t_1 $Z^W(t_1) = U(t_1)$ and $Z^A(t_1-) = Z^A(t_1)$. Also $Z(t_1) = Z(t_1-) + U(t_1)$. But we know $Z(t_1-) = Z^A(t_1-)$. Thus

$$\begin{aligned} Z(t_1) &= Z(t_1-) + U(t_1) \\ &= Z^A(t_1) + Z^W(t_1). \end{aligned}$$

Suppose, on the other hand, that $T_o \leq t_1$. Then $Z^A(t_1-) = 0 = Z(t_1-)$ and $Z^W(t_1) = 0$. Also

$$Z^A(t_1) = U(t_1) = Z(t_1) \quad \text{and} \quad Z^W(t_1) = 0,$$

since the arriving input is put directly into the store, giving

$$Z(t_1) = U(t_1) = Z^A(t_1) + Z^W(t_1).$$

Thus

$$Z(t) \leq Z^A(t) + Z^W(t) \qquad \text{for } t \leq t_1$$

Now suppose we know that $Z^A(t) + Z^W(t) \geq Z(t)$ for all $t \leq t_k$. We want to prove that

$$Z^A(t) + Z^W(t) \geq Z(t) \qquad \text{for } t \leq t_{k+1}$$

For definiteness, let $U(t_l)$, $U(t_{l+1})$,... $U(t_m)$ be the inputs stored in the warehouse and define

$$T_{k,l} = T_k = \inf \{t \geq t_k: Z^A(t) = 0\}$$

$$T_{k,l+1} = \inf \{t > T_{k,l}: Z^A(t) = 0\}$$

There are a number of cases to consider.

Case I: $t_k < t \leq t_{k+1} \leq T_k$

Suppose $t_k < t < t_{k+1}$

Recall that

$$t-t_k = \int_{Z(t-)}^{Z(t_k)} \frac{du}{r(u)} = \int_{Z^A(t-)}^{Z^A(t_k)} \frac{du}{r(u)}$$

From Lemma 2.1.2 we have

$$\int_{Z^A(t-)}^{Z^A(t_k)} \frac{du}{r(u)} \geq \int_{Z^A(t-)+Z^W(t_k)}^{Z^A(t_k)+Z^W(t_k)} \frac{du}{r(u)}$$

and we know by assumption that $Z(t_k) \leq Z^A(t_k) + Z^W(t_k)$

Thus

$$\int_{Z^A(t-)+Z^W(t_k)}^{Z^A(t_k)+Z^W(t_k)} \frac{du}{r(u)} \geq \int_{Z^A(t-)+Z^W(t_k)}^{Z(t_k)} \frac{du}{r(u)}$$

which yields

$$\int_{Z(t-)}^{Z(t_k)} \frac{du}{r(u)} \geq \int_{Z^A(t-)+Z^W(t_k)}^{Z(t_k)} \frac{du}{r(u)}$$

For the inequality to hold we must have $Z(t-) \leq Z^A(t-) + Z^W(t_k)$.

Since $t \neq T_k$, $Z^A(t-) = Z^A(t)$.

Also, for $t \in (t_k, t_{k+1})$ $Z(t-) = Z(t)$ and $Z^W(t_k) = Z^W(t)$.

Thus we have

$$Z(t) \leq Z^A(t) + Z^W(t) \qquad t_k < t < t_{k+1}$$

At $t = t_{k+1}$

$$Z(t_{k+1}) = Z(t_{\overline{k+1}}) + U(t_{k+1})$$

$$\leq Z^A(t_{\overline{k+1}}) + Z^W(t_{\overline{k+1}}) + U(t_{k+1})$$

$$= Z^A(t_{k+1}) + Z^W(t_{k+1})$$

The latter is true, since if $t_{k+1} < T_k$ then

$$Z^A(t_{k+1}) = Z^A(t_{\overline{k+1}})$$

and $$Z^W(t_{k+1}) = Z^W(t_{\overline{k+1}}) + U(t_{k+1})$$

<u>or</u> if $t_{k+1} = T_k$

$$Z^A(t_{k+1}) = U(t_{k+1})$$

and $$Z^W(t_{\overline{k+1}}) = Z^W(t_{k+1})$$

since if an input arrives just as the alternative store empties, this input goes into the store. Thus we have

$$Z(t) \leq Z^A(t) + Z^W(t) \qquad \text{for } t_k \leq t \leq t_{k+1} \leq T_k$$

<u>Case II</u>: $t_k < t \leq T_k < t_{k+1}$

Suppose $t < T_k$. As we did for Case I we can get

$$\int_{Z(t-)}^{Z(t_k)} \frac{du}{r(u)} = \int_{Z^A(t-)}^{Z^A(t_k)} \frac{du}{r(u)} \geq \int_{Z^A(t-)+Z^W(t_k)}^{Z^A(t_k)+Z^W(t_k)} \frac{du}{r(u)} \geq \int_{Z^A(t-)+Z^W(t_k)}^{Z(t_k)} \frac{du}{r(u)}$$

which implies

$$Z(t-) \leq Z^A(t-) + Z^W(_k).$$

Again since $t \in (t_k, t_{k+1})$, $Z(t-) = Z(t)$ and $Z^A(t-) = Z^A(t)$. Also since $t < T_k$, $Z^W(t_k) = Z^W(t)$, which gives

$$Z(t) \leq Z^A(t) + Z^W(t).$$

For $t = T_k$, since $Z^A(T_k) = U(t_1)$ and $Z^W(T_k) = Z^W(T_k^-) - U(t_1)$

we have

$$Z^A(T_k^-) + Z^W(T_k^-) = Z^A(T_k) + Z^W(T_k)$$

Thus

$$Z(T_k) = Z(T_k-) \leq Z^A(T_k^-) + Z^W(T_k^-) = Z^A(T_k) + Z^W(T_k)$$

<u>Case III</u>: $t_k < T_{k,1} < \dots < T_{k,j} < t < T_{k,j+1} < \dots < T_{k,l} < t_{k+1} < T_{k,l+1}$

There can be one or more times before t_{k+1} where the store in the warehouse system will empty. We look at the interval $T_{k,1} < t \leq T_{k,2}$ which is typical of the others. Again, for $t < T_{k,2}$

$$\int_{Z(t-)}^{Z(T_{k,1})} \frac{du}{r(u)} = \int_{Z^A(t-)}^{Z^A(T_{k,1})} \frac{du}{r(u)} \geq \int_{Z^A(t-)+Z^W(T_{k,1})}^{Z^A(T_{k,1})+Z^W(T_{k,1})} \frac{du}{r(u)} \geq \int_{Z^A(t-)+Z^W(T_{k,1})}^{Z(T_{k,1})} \frac{du}{r(u)}$$

Thus we have

$$Z(t-) \leq Z^A(t-) + Z^W(T_{k,1}).$$

But $Z(t) = Z(t-)$, $Z^W(T_{k,1}) = Z^W(t)$ and $Z^A(t-)=Z^A(t)$, since no inputs or emptiness occur at t. Thus

$$Z(t) \leq Z^A(t) + Z^W(t) \quad \text{for} \quad T_{k,1} < t < T_{k,2}$$

For $t = T_{k,2}$ we have $Z^A(T_{k,2}) + Z^W(T_{k,2}) = Z^A(T_{k,2}^-) + Z^W(T_{k,2}^-)$

which means

$$Z(T_{k,2}) = Z(T_{k,2}^-) \leq Z^A(T_{k,2}^-) + Z^W(T_{k,2}^-) = Z^A(T_{k,2}) + Z^W(T_{k,2})$$

So for $T_{k,1} < t \leq T_{k+1}$

$$Z(t) \leq Z^A(t) + Z^W(t).$$

Clearly, it will also be the case that

$$Z(t) \leq Z^A(t) + Z^W(t) \quad \text{for} \quad T_{k,j} < t < T_{k,j+1}$$

We now look at the last case.

<u>Case IV</u>: $T_{k,1} < t \leq t_{k+1} \leq T_{k,1+1}$

Suppose $T_{k,1} < t < t_{k+1}$. As usual

$$\int_{Z(t-)}^{Z(T_{k,1})} \frac{du}{r(u)} = \int_{Z^A(t-)}^{Z^A(T_{k,1})} \frac{du}{r(u)} \geq \int_{Z^A(t-)+Z^W(T_{k,1})}^{Z^A(T_{k,1})+Z^W(T_{k,1})} \frac{du}{r(u)} \geq \int_{Z^A(t-)+Z^W(T_{k,1})}^{Z(T_{k,1})} \frac{du}{r(u)}$$

which implies

$$Z(t-) \leq Z^A(t-) + Z^W(T_{k,1}).$$

For $t < t_{k+1}$ we have $Z(t-) = Z(t)$ and $Z^W(T_{k,1}) = Z(t)$.

Also, since $t \in (T_{k,l}, T_{k,l+1})$ we have $Z^A(t-) = Z^A(t)$. This gives

$$Z(t) \leq Z^A(t) + Z^W(t).$$

For $t = t_{k+1}$

$$Z(t_{k+1}) = Z(t_{\overline{k+1}}) + U(t_{k+1}) \leq Z^A(t_{\overline{k+1}}) + Z^W(t_{\overline{k+1}}) + U(t_{k+1})$$

But we know that either $Z^A(t_{\overline{k+1}}) = Z^A(t_{k+1})$ and $Z^W(t_{k+1}) = Z^W(t_{\overline{k+1}}) + U(t_{k+1})$ if $t_{k+1} < T_{k,l+1}$, or if $t_{k+1} = T_{k,l+1}$ then $Z^A(t_{\overline{k+1}}) = 0$, $Z^W(t_{k+1}) = Z^W(t_{\overline{k+1}})$, $Z^A(t_{k+1}) = U(t_{k+1})$. In either case

$$Z^A(t_{\overline{k+1}}) + Z^W(t_{\overline{k+1}}) + U(t_{k+1}) = Z^A(t_{k+1}) + Z^W(t_{k+1})$$

Thus

$$Z(t) \leq Z^A(t) + Z^W(t) \quad \text{for} \quad T_{k,l} < t \leq t_{k+1} \leq T_{k,l+1}.$$

With these four cases we have shown that

$$Z(t) \leq Z^A(t) + Z^W(t) \quad \text{for} \quad t \leq t_{k+1}$$

and by induction this then holds for all t.

Recall that we denote the busy time and empty time by b(t) and e(t) respectively. For the alternative warehouse model we will use $b^A(t)$ and $e^A(t)$. Since $Z(t) \leq Z^A(t) + Z^W(t)$ we have $e(t) \geq e^A(t)$ and $b(t) \leq b^A(t)$.

We call $Y(U(t_j)) = \int_0^{U(t_j)} \frac{du}{r(u)}$ the standard busy time. We can think of Y as the amount of busy time associated with the input $U(t_j)$ in the warehouse model. We note that the busy time associated with $U(t_j)$ in the original model will be no bigger than this standard busy time.

For the warehouse model we have the inequality

(11) $E\left[b^A(n)\right] \leq E\left[Y\ (Z(0))\right] + n\mu_y$

The first term on the right hand side of (11) represents the standard busy time associated with the initial content Z(0). The second term represents the standard busy time from all inputs to arrive at the warehouse system in (0,n). The inequality indicates that possibly all inputs arriving in (0,n) are not used up during that time (i.e. possibly $Z^W(n) > 0$) so the right hand side of (11) may be an overestimate.

Since $e^A(t) = t - b^A(t)$ we have

$$\begin{aligned} E\left[e(n)\right] &\geq E\left[e^A(n)\right] \\ &= n - E\left[b^A(n)\right] \\ &\geq n - n\mu_y - E\left[Y(Z(0))\right] \end{aligned}$$

Then

$$E\frac{\left[e(n)\right]}{n} \geq 1 - \mu_y - E\frac{\left[Y(Z(0))\right]}{n}$$

and since by assumption $1 - \mu_y > 0$

$$\lim_{n\to\infty} E\frac{\left[e(n)\right]}{n} \geq 1 - \mu_y > 0$$

<u>Lemma 2.2.2</u>: With a storage process as defined in Theorem 2.2, there exists τ, $0 \leq \tau < 1$ such that ε_τ will recur with probability 1, and the mean recurrence time μ_τ is finite.

<u>Proof</u>: Let $\zeta_t = \begin{cases} 1 & \text{if } Z(t) = 0 \\ 0 & \text{if } Z(t) > 0 \end{cases}$

Then $\int_0^n \zeta_t \, dt = e(n)$

and $I(n) = E\left[e(n)\right]$

$$= \int_0^n P\{Z(t) = 0\}\, dt$$

$$= \int_0^n \sum_{j=o}^{n-1} P\{Z(j+\tau) = 0\}\, d\tau$$

From Lemma 2.2.1 we know that $\lim\limits_{n\to\infty} \frac{1}{n} I(n) > 0$.

Then

$$\lim_{n\to\infty} \int_0^1 \sum_{j=o}^{n-1} P\{Z(j+\tau) = 0\}\, d\tau > 0$$

We can think of the event ε_τ as a delayed recurrent event where ε_τ stands for "a success occurs" (system is empty at $t=n+\tau$) in a sequence of Bernoulli trials. Pursuing this (in the manner of Feller) we note then that

$$U_\tau(n) = \sum_{j=o}^{n-1} P\{Z(j+\tau) = 0\}$$

is the expected number of renewals in $[0,n]$. What can we say about $U_\tau(n)/n$?

If $\mu_\tau = \infty$ or the probability ε_τ recurs is less than 1 then

$$\lim_{n\to\infty} \frac{U_\tau(n)}{n} = 0.$$

If ε_τ recurs with probability one and the first occurrence of ε_τ has probability $p(\tau) > 0$ then

$$\lim_{n\to\infty} \frac{U_\tau(n)}{n} = \frac{p(\tau)}{\mu_\tau}$$

In any case, it is always true that $U_\tau(n)/n$ coverges to a limit as n goes to infinity. We will call this limie $\Theta(\tau)$. Since $\Theta(\tau) \leq \mu_\tau^{-1}$ we have, by dominated convergence, that

$$\lim_{n\to\infty} \frac{I(n)}{n} = \lim_{n\to\infty} \int_0^1 \frac{U_\tau(n)}{n}\, d\tau$$

$$= \int_0^1 \Theta(\tau)\, d\tau$$

$$> 0$$

Thus we must have that $\Theta(\tau) > 0$ on a set of positive Lebesgue measure. Let $\Theta = \{\tau: \Theta(\tau) > 0\}$. For $\tau \varepsilon \Theta$ we have that

$$\lim_{n\to\infty} \frac{U_\tau(n)}{n} > 0$$

which implies that $\mu_\tau < \infty$, ε_τ recurs with probability one and $p(\tau) > 0$.

Lemma 2.2.3: For a storage system as described in Theorem 2.2, if $\tau \varepsilon \Theta$ then the first occurrence of ε_τ is certain.

Proof: Suppose we have two storage systems, $Z_1(t) = Z(t,\omega,\xi_1,r)$ and $Z_2(t) = Z(t,\omega,\xi_2,r)$ which are identical except for initial store content, and $\xi_2 > \xi_1 = 0$. By Lemma 2.1.1 we know that $Z_1(t) \leq Z_2(t)$ for all t. If there is a $t_o = t_o(\xi_2,\omega)$ such that $Z_2(t_o) = 0$ then $Z_1(t_o) = 0$ and for $t \geq t_o$ $Z_1(t) = Z_2(t)$. We will prove that there must exist such a t_o. Suppose not. Then $e_2(t) = 0$ for all t. But this implies that $I_2(t) = E_2(e(t)) = 0$, a contradiction as long as $\mu_y < 1$ (Lemma 2.2.1). Hence there must exist a t_o.

Suppose now that we have two processes, $Z_3(t) = Z(t,\omega_1,\xi_3,r)$ and $Z_4(t) = Z(t,\omega_1,\xi_4,r)$ such that $Z_3(\tau) = 0$ and $Z_4(\tau) = \xi$ for some $\tau \varepsilon \Theta$. Construct two new processes with τ as the new origin: $Z_3^*(t) = Z_3(t+\tau)$ and $Z_4^*(t) = Z_4(t+\tau)$. We know there exists a $t_o^*(\xi,\omega_1)$ such that $Z_3^*(t) = Z_4^*(t)$ for $t \geq t_o^*(\xi,\omega_1)$. By Lemma 2.2.2 we know that for some $k \geq t_o^*(\xi,\omega_1)$, $Z_3(k+\tau) = 0$, since for $\tau \varepsilon \Theta, \varepsilon_\tau$ recurs with probability one. But $Z_3(k+\tau) = Z_3^*(k) = Z_4^*(k)$ since $k \geq t_o^*(\xi,\omega_1)$, and $Z_4^*(k) = Z_4(k+\tau)$, i.e. $Z_4(k+\tau) = 0$. Thus we have shown that if we have a storage system which meets the requirements of the Lemma, and $\tau \varepsilon \Theta$, then ε_τ is certain.

<u>Proof of Theorem 2.2 continued</u>: Choose $\tau \varepsilon \Theta$. From Lemma 2.2.3 we know ε_τ is certain. To show that $\sum_{n=o}^{\infty} \phi_A(n+\xi)(1 - F(n+\xi)) < \infty$ note that

$$\sum_{n=o}^{\infty} \phi_A(n+\xi)(1 - F(n+\xi)) \leq \sum_{n=o}^{\infty} (1-F(n+\xi)) \leq \mu_1 < \infty$$

by Lemma 2.2.2. Thus in conclusion, by Theorem 3 of Smith (1955)

$$\lim_{n\to\infty} P\{Z(n+\xi) \ \varepsilon A \mid Z(0)\} = \frac{1}{\mu_1} \sum_{n=o}^{\infty} \phi_A(n+\xi)(1 - F(n+\xi))$$

The next result follows directly from Theorem 2.2, but now the release rule is not constrained to be nondecreasing. Instead, we assume that the release rule is always bounded away from zero.

<u>Theorem 2.3</u>: Let $Z(t)$ be the content of a Type A storage system with general periodic input and release rule $r(x)$ such that

$$r(x) \geq \alpha > \sum_{j=1}^{p} \int_o^{\infty} x \, d\, G_j(x) + \int_o^1 \int_o^{\infty} x \, B(dx,u)\, \lambda(u)\, du \quad \text{for each}$$

$x > 0$. Then there exists a quasi-limiting distribution of store content which is independent of the initial content distribution.

Proof: Define two Type A storage processes with g.p.i. $Z_1(t) = Z(t,\omega,\xi,r_1)$ and $Z_2(t) = Z(t,\omega,\xi,r_2)$ where $r_1(x)$ is as described in the theorem and $r_2(x) = \alpha$, $x > 0$. We will denote functions of the i^{th} process with subscript i, i+1,2. Let the regenerative event $\varepsilon_i(\tau)$ be the event which occurs at time n+1 if $Z_i(n+\tau) = 0$, $\tau \varepsilon \Theta$, i=1,2.

Since $r_1(x) \geq r_2(x)$ we have $Z_1(t) \leq Z_2(t)$. In particular, $Z_2(t) = 0$ implies $Z_1(t) = 0$. Thus if we can show that $\varepsilon_2(\tau)$ is certain and $\mu_2 < \infty$ we will have shown that $\varepsilon_1(\tau)$ is certain and $\mu_1 < \infty$. From Lemmas 2.2.1 - 2.2.3 we know that $\varepsilon_2(\tau)$ is certain and $\mu_2 < \infty$ if we can show that

$$\sum_{l=1}^{p} \int_0^\infty Y_2(u)\, G_l(du) + \int_0^1 \int_0^\infty Y_2(u)\, B(du,t)\, \lambda(t)\, dt < 1.$$

This follows from the definition of Y(u). Since

$$Y_2(u) = \int_0^u \frac{dx}{r_2(x)} = \frac{1}{\alpha} \int_0^u dx = \frac{1}{\alpha} U$$

we have that

$$\sum_{l=1}^{p} \int_0^\infty Y_2(u)\, G_l(du) + \int_0^1 \int_0^\infty Y_2(u)\, B(du,t)\, \lambda(t)\, dt$$

$$= \sum_{l=1}^{p} \int_0^\infty \frac{1}{\alpha} u\, G_j(du) + \int_0^1 \int_0^\infty \frac{1}{\alpha} u\, B(du,t)\, \lambda(t)\, dt$$

$$< \frac{1}{\alpha} \cdot \alpha = 1$$

by the assumption of the Lemma. From Theorem 3 of Smith (1955) we can conclude that a quasi-limiting distribution exists.

2.5 A Limit Theorem for Type B Processes with General Periodic Input

In looking at a Type A storage process we have the advantage of being able to focus on the times the store is empty, a convenient event which is regenerative in nature. The limit theorems of Harrison and Resnick (1976) require assumption (iv) to rule out any process which cannot empty for this very purpose. But Type B processes are quite common and one would hope that results similar to Theorem 2.2 or 2.3 should exist. And indeed such is the case.

The next theorem is for a Type B process with periodic input--a partner to Theorem 2.2. The fact that a Type B process, while not emptying, can have content arbitrarily close to zero is used to set up a "false bottom" near zero where Theorem 2.2 can apply.

Theorem 2.4: Suppose $Z(t)$ is the content of a Type B storage system with general periodic input and nondecreasing release rule $r(x)$. Define

$$Y(u,\varepsilon) = \int_{\varepsilon}^{u} \frac{dx}{r(x)}.$$

If, for every ε

$$\mu_\varepsilon = \sum_{l=1}^{p} \int_0^\infty Y(u,\varepsilon)\, G_l(du) + \int_0^1 \int_0^\infty Y(u,\varepsilon)\, B(du,t)\, \lambda(t)\, dt < 1$$

then the store content has a quasi-limiting distribution which is independent of the initial content distribution.

Proof: Let $Z(t) = Z(t,\varepsilon,\xi,r)$ be a Type B process as described in the theorem. In addition, define a sequence of storage processes $Z_n(t) = Z(t,\omega,\xi,r_n)$ which have the same input ω and initial input ξ as the $Z(t)$ process, but release rule $r_n(x)$, where

$$r_n(x) = \begin{cases} r(x) & x > 1/n \\ 0 & x \le 1/n \end{cases}$$

Note that each Z_n- process is a Type A process with "bottom" 1/n. Since $\mu_{1/n} < 1$ the assumptions of Theorem 2.2 are satisfied for each Z_n- process. Thus we can conclude that each Z_n- process will have a quasi-limiting distribution; i.e., if $H_n(x,t) = P\{Z_n(t) \le x\}$ then

$\lim_{k\to\infty} H_n(x,k+\tau) = \tilde{H}_n(x,\tau)$ where $\tilde{H}_n$ is the quasi-limiting distribution.

Since $\{r_n\}$ is a nondecreasing sequence, by Lemma 2.1.3 $\{Z_n(t)\}$ is nonincreasing. Thus $\{H_n(x,t)\}$ is a nondecreasing sequence of distribution functions for each t. This implies that $\{\tilde{H}_n(x,\tau)\}$ is also a nondecreasing sequence, since if m>n implies

$$H_n(x,k+\tau) \le H_m(x,k+\tau)$$

then

$$\tilde{H}_n(x,\tau) = \lim_{k\to\infty} H_n(x,k+\tau) \le \lim_{k\to\infty} H_m(x,k+\tau) = \tilde{H}_m(x,\tau).$$

Thus we know that a limit of the distribution functions exists. Let

$$\tilde{H}(x,\tau) = \lim_{n\to\infty} \tilde{H}_n(x,\tau).$$

Since $\{\tilde{H}_n(x,\tau)\}$ is a nondecreasing sequence we have $\tilde{H}_1(x,\tau) \le \tilde{H}_n(x,\tau)$ for all n. $\tilde{H}_1(x,\tau)$ is a distribution function, so $\lim_{x\to\infty} \tilde{H}_1(x,\tau) = 1$. Thus for every ε there is an A_ε such that $\tilde{H}_1(A_\varepsilon,\tau) > 1-\varepsilon$. Then we know that $\tilde{H}_n(A_\varepsilon,\tau) > 1-\varepsilon$ for all n. This means that $\{\tilde{H}_n(x,\tau)\}$ is a tight sequence, since for all n

$$\tilde{H}_n(A_\varepsilon,\tau) - \tilde{H}_n(-A_\varepsilon,\tau) = \tilde{H}_n(A_\varepsilon,\tau) > 1 - \varepsilon.$$

Thus we can conclude that $\tilde{H}(x,\tau)$ is a distribution function and $\tilde{H}_n(x,\tau)$ converges weakly to $\tilde{H}(x,\tau)$.

Next we show that for each n, $0 \leq Z_n(t) - Z(t) \leq \frac{1}{n}$, for all t. Let ε_τ^n be the event: ε_τ^n occurs at t=k+1 if $Z_n(k+\tau) = 1/n$, $0 \leq \tau < 1$. Since $Z_n(t)$ is the content of a Type A process and $\mu_\varepsilon < 1$ we know, from Lemma 2.2.3 that for some τ, ε_τ is certain. Thus if we let T_n be the first time $Z_n(t) = \frac{1}{n}$ then T_n is a proper random variable. For $t < T_n$, $Z(t) = Z_n(t)$. Let t_1 be the time of the first input after T_n. For $T_n < t < t_1$ we have $Z_n(t) = \frac{1}{n}$ and $Z(t) < \frac{1}{n}$, so $0 \leq Z_n(t) - Z(t) \leq \frac{1}{n}$ for $t < t_1$. At t_1, there is an input of size U, say. Then $Z_n(t_1) = U + Z_n(t_1-)$ and $Z(t_1) = U + Z(t_1-)$. So

$$0 \leq [Z_n(t_1-) + U] - [Z(t_1-) + U] \leq \frac{1}{n} .$$

But this is equivalent to

$$0 \leq Z_n(t_1) - Z(t_1) \leq \frac{1}{n} .$$

Thus for $t \leq t_1$

(12) $$0 \leq Z_n(t) - Z(t) \leq \frac{1}{n} .$$

Since (12) holds both between inputs and directly after an input, we can conclude that it holds for all t.

From (12) it must hold that for every n

(13) $$P\{Z_n(t) \leq x\} \leq P\{Z(t) \leq x\} \leq P\{Z_n(t) \leq x + \frac{1}{n}\} .$$

Let $H(x,t) = P\{Z(t) \leq x\}$. Then (13) can be rewritten

$$H_n(x,t) \le H(x,t) \le H_n(x + \frac{1}{n}, t) \quad \text{for all } t.$$

This implies

$$(14) \qquad \tilde{H}_n(x,\tau) \le \varliminf_{k\to\infty} H(x,k+\tau) \le \varlimsup_{k\to\infty} H(x,k+\tau) \le \tilde{H}_n(x + \frac{1}{n},\tau)$$

For any ε, let n be large enough that $\varepsilon > \frac{1}{n}$. Then

$\tilde{H}_n(x + \frac{1}{n},\tau) \le \tilde{H}_n(x+\varepsilon,\tau)$. Letting n go to infinity in (14) we get

$$\tilde{H}(x,\tau) \le \varliminf_{k\to\infty} H(x,k+\tau) \le \varlimsup_{k\to\infty} H(x,k+\tau) \le \tilde{H}(x+\varepsilon,\tau)$$

Letting ε approach zero we have

$$\tilde{H}(x,\tau) \le \varliminf_{k\to\infty} H(x,k+\tau) \le \varlimsup_{k\to\infty} H(x,k+\tau) \le \tilde{H}(x+0,\tau)$$

At every continuity point of $\tilde{H}$ we have $\lim_{k\to\infty} H(x,k+\tau) = \tilde{H}(x,\tau)$.

Thus we have shown that $H(x,k+\tau)$ converges weakly to $\tilde{H}(x,\tau)$, the desired conclusion.

2.6 A Limit Theorem for a Homogeneous Storage System

We close the chapter with a theorem concerning homogeneous storage processes. The result was originally developed as a response to the paper by Harrison and Resnick (1976), whose conditions for the existence of a limiting distribution are not at all obvious or intuitive. Also, it appeared that the conditions would often be difficult, if not impossible to verify. In addition, the paper unnecessarily eliminated a rich class of processes--Type B processes. It seemed apparent that some other level than the zero level could be used as a basis for a

regenerative event, allowing Type B processes. For these reasons, Theorem 2.5 was developed.

<u>Theorem 2.5</u>: Let $Z(t)$ be the content of a storage system with homogeneous Compound Poisson input, intensity λ, and release rule $r(x)$ which satisfies the following criterion:

There exists a level c such that

$$r(x) \geq \alpha \quad \text{for } x > c$$

$$r(x) \leq M_c \quad \text{for } x \leq c$$

for some constants α and M_c.

If the first moment of the input size distribution (β_1) is finite and $\alpha > \lambda\, \beta_1$ then there exists a limiting distribution of store content that is independent of initial conditions.

<u>Proof</u>: For the store content $Z_1(t) = Z(t,\omega,\xi,r_1)$ as described above, let us say ε_1^A occurs at t if $Z_1(t) = c$ and $Z_1(t-) > c$. (i.e. $Z_1(t)$ crosses c from above at time t). We do not consider ξ=c an occurrence of ε^A.

Clearly ε_1^A is a regenerative event. If we can show the following three criteria are satisfied then the desired limiting distribution will exist.

(B1) ε_1^A is certain

(B2) The time between successive occurrence of ε_1^A has finite expectation

(B3) For some useful class of sets C, $\phi_c(t)(1-F(t))$ is of bounded variation in every finite interval, $c \in C$, where $\phi_c(t)(1-F(t)) = P\{Z_1(T_\xi^A + t)\varepsilon c, t_1 > t \mid \xi, T_\xi^A\}$ and T_ξ^A is the time of the first occurrence of ε_1^A.

If c=0 then we have $r_1(x) \geq \alpha > \lambda\beta_1$ for $x > 0$. Define $Z_3(t) = Z(t,\omega,\xi,r_3)$ where $r_3(x) = \alpha \leq r_1(x)$. From Lemma 2.1.1 $Z_1(t) \leq Z_3(t)$. Thus if $Z_3(t) = 0$ then $Z_1(t) = 0$. From standard storage theory (see Appendix) we know that the Z_3 - process will empty with probability one and the expected time between emptying is finite. Thus we have (B1) and (B2) satisfied for the case c=0.

To prove (B1)-(B2) for $c > 0$ we will need to define some additional events and random variables. Let us say ε_1^B occurs at t if $Z_1(t-)<c$ and $Z_1(t)\geq c$. Let U_1 be the time from an ε_1^A to the next occurrence of ε_1^B, V_1 be the time from an ε_1^B to the next occurrence of ε_1^A and W_1 be the time an ε_1^B to the next occurrence of ε_1^A. Define $Z_2(t)=Z(t,\omega,\xi,r_2)$ where $r_2(x)=M_c$, $x > 0$. All the above events and random variables apply to the Z_2 - and Z_3 - process (defined above), denoted by appropriate subscripts. Note that the Z_2 and Z_3 systems have homogeneous Compound Poisson input and constant release rule. Results on such processes are well known. In particular, we know that $Z_2(t)$ and $Z_3(t)$ are type A processes and can empty. Thus for $Z_2(t)$ and $Z_3(t)$ we can introduce the event ε_j which occurs at t if $Z_j(t)=0$ and $Z_j(t-)>0$, j=2,3. The random variable X_j will be the length of the busy cycle in the Z_j - process, j=2,3.

To begin, we show that ε_1^A is certain. Suppose $\xi > c$. For $t \leq T_\xi^A$ we have $Z_3(t) \geq Z_1(t)$, since $r_1(x) \geq \alpha = r_3(x)$. Thus ε_1^A will be certain if ε_3^A is, and ε_3^A will be certain if ε_3 is. But $\lambda\,\beta_1 < r_3(x) = \alpha$ implies, by standard storage theory, that ε_3 is certain. Thus we have that ε_1^A is certain if $\xi > c$.

Let us next suppose $\xi \leq c$. The event ε_1^A will be certain if ε_1^B is certain and V_1 is a proper random variable. Let T_ξ^B be the time of the

first occurrence of ε_1^B. For $t \leq T_2^B$ we have $Z_2(t) \leq Z_1(t)$. This is true, since while $Z_1(t) < c$, $r_1(x) \leq M_c = r_2(x)$. Thus ε_1^B will occur before ε_2^B does, which implies that ε_1^B is certain if ε_2^B is.

Let

$p = P\{Z_2(t)$ jumps above c during a busy cycle of the Z_2 - process$\}$

$\tilde{p} = P\{Z_2(t)$ jumps above c during the initial busy cycle of the Z_2-process$\}$

$q = 1 - p, \quad \tilde{q} = 1 - \tilde{p}$

If we assume $p > 0$ then

$$P\{T_\varepsilon^B < \infty\} = P\{\varepsilon_2^B \text{ first occurs during the initial busy cycle}\}$$

$$+ \sum_{k=1}^{\infty} P\{\varepsilon_2^B \text{ first occurs during the } k^{th} \text{ busy cycle}\}$$

$$= \tilde{p} + \tilde{q}\, p + \tilde{q}\, qp + \tilde{q}\, (q)^2\, p + \ldots$$

$$= \tilde{p} + \tilde{q}\, p\, (1 + q + q^2 + \ldots)$$

$$= \tilde{p} + \tilde{q}\, p/p$$

$$= 1$$

i.e., if $p>0$ then the first occurrence of ε_2^B is certain. Now, in any interval of length y we know that the output of the Z_2 - process must be less than or equal to $M_c y$, since $r_2(x) = M_c$. We know that with Compound Poisson input there is a positive probability of input greater than $c + M_c\, y$. Thus we have $p>0$ and ε_2^B is certain, implying ε_1^B is certain.

Note that $Z_1(T_\xi^B) > c$. Thus showing that V_1 is a proper random variable is equivalent to showing that ε_1^A is certain if $\xi_1 = Z_1\,(T_\xi^B)$, which we have already shown. Hence we have that ε_1^A is certain.

To show that (B2) is satisfied we need the following lemma.

Lemma 2.5.1: Let $Z_1(t)$ be the content of a storage system as described in Theorem 2.5. Let ε_1^A occur at t if $Z_1(t)=c$ and $Z_1(t-)>c$, and W_1 be the time between successive occurrences of ε_1^A. If $\beta_2 < \infty$ then $E(W_1) < \infty$.

Proof: We employ the same processes and random variables as in the proof of Theorem 2.5. Define further the following random variables:

η is the length of a busy cycle of $Z_2(t)$ during which the store content does not cross level c during the busy cycle.

ζ is the length of a busy cycle of $Z_2(t)$ during which the store content does cross level c during the busy cycle.

Since $W_1 = U_1 + V_1$ we need to look at U_1 and V_1. Suppose ξ=c. Then $U_1 \leq U_2$. This is true since when $Z_1(t) \leq c, r_1(x) \geq r_2(x)$, so $Z_2(t)$ will take at least as long as $Z_1(t)$ to rise above c. Now

$$U_2 \leq \eta_1 + \eta_2 + \ldots + \eta_N + \zeta = \Theta$$

where N, a random variable, is the number of times the process does not cross level c during successive busy cycles, before finally crossing c. Let $S_N = \eta_1 + \eta_2 \ldots + \eta_N$, so that $\Theta = S_N + \zeta$, a random sum. Note that η, ζ and N are mutually independent. Using Walds equation

$$E(\Theta) = E(S_N) + E(\zeta)$$

$$= E(N)E(\eta) + E(\zeta)$$

We also have

$$E(\Theta^2) = E(S_N^2) + 2E(\zeta S_N) + E(\zeta^2)$$

Since ζ, η and N are independent

$$E(\zeta S_N) + E(\zeta)E(N)E(\eta)$$

What is $E(S_N^2)$?

$$E(S_N^2) = E\left[E(S_N^2 \mid N)\right]$$

$$= \sum_{k=o}^{\infty} P(N=k)E(S_k^2)$$

$$= \sum_{k=o}^{\infty} P(N=k)\{\sum_{j=1}^{k} E(\eta_j^2) + 2 \sum_{j=1}^{k=1} \sum_{l=j+1}^{k} E(\eta_j\eta_l)\}$$

$$= \sum_{k=1}^{\infty} P(N=k) \{kE(\eta^2) + k(k-1)\left[E(\eta)\right]^2\}$$

$$= E(\eta^2) \sum_{k=1}^{\infty} kP(N=k) + (E\eta)^2 \{\sum_{k=1}^{\infty} k^2P(N=k) - \sum_{k=1}^{\infty} kP(N=k)\}$$

$$= E(\eta^2)E(N) + (E\eta)^2 \left[E(N^2) - E(N)\right]$$

Thus

$$E(\Theta^2) = E(\eta^2)E(N) + (E\eta)^2 \left[E(N^2) - E(N)\right] + 2E(\zeta)E(N)E(\eta) + E(\zeta^2)$$

We know that $P(N=k) = q^k p$, where p and q are as defined in the proof of Theorem 2.5. Thus $E(N) = q/p < \infty$ if $p>0$. But in the proof of Theorem 2.5 we showed that $p>0$. Both $E(\eta)$ and $E(\zeta)$ will be finite if $E(X_2) < \infty$ and $p>0$, since $E(X_2) = q\ E(\eta) + p\ E(\zeta)$. By standard storage theory results, $E(X_2) < \infty$ if $\beta_1 < \infty$ and $\lambda\ \beta_1 < M_c$, both of which are true by assumption. Similarly, $E(\eta^2)$ and $E(\zeta^2)$ will be finite if $\beta_2 < \infty$ and $\lambda\ \beta_1 < M_c$. Thus we have that for $\xi=c$

$$E(U_1) \leq E(U_2) \leq E(\Theta) < \infty$$

$$E(U_1^2) \leq E(U_2^2) \leq E(\Theta^2) < \infty$$

Suppose $\xi \neq c$ and define $Z_1^*(t) = Z_1(t+T_\xi^A,\omega,\xi,r_1)$. We know that $E(U_1^*) < \infty$, where U_1^* is as defined earlier, but with reference to the Z_1^* - process, rather than $Z_1(t)$. Note that $Z_1^*(U_1^*) > c$ implies that $Z_1(T_\xi^A + U_1^*) > c$, i.e. $U_1 \leq U_1^*$. Thus we have that $E(U_1) \leq E(U_1^*)$ for all ξ.

We still need to consider V_1. Suppose $\xi > c$. We have shown earlier that $V_1 < V_3$, and clearly we know that $V_3 < \zeta$. Thus we have that

$$E(V_1) < E(\zeta)$$

$$E(V_1^2) < E(\zeta^2)$$

and we know that these both will be finite since $\beta_1 < \infty, \beta_2 < \infty$ and $\lambda\ \beta_1 < \alpha$.

Next suppose we drop the assumption that $\xi = c$. Define $Z_1^{**}(t) = Z_1(t+T_\xi^B,\omega,\xi,r_1)$. Then $Z_1^{**}(0) > c$. We know that $E(V_1^{**}) < \infty$ where V_1^{**} is as defined earlier, but with reference to the Z_1^{**} - process. Note that $Z_1^{**}(V_1^{**}) \leq c$, which implies that $Z_1(V_1^{**} + T_\xi^B) \leq c$, i.e. $V_1 \leq V_1^{**}$. Thus we have $E(V_1) \leq E(V_1^{**}) < \infty$ for all ξ.

Since $W_1 = U_1 + V_1$ we have $E(W_1) = E(U_1) + E(V_1) < \infty$ and since U_1 and V_1 are independent

$$E(W_1^2) = E(U_1^2) + E(V_1^2) + 2E(U_1)E(V_1) < \infty$$

and the lemma is proved.

To finish the proof of the theorem, we must prove (B3). This can be done using the following adaptation of Lemma 2 of Smith (1955).

Lemma 2 (Smith): If (i) $I = (t_o+a,\ t_o+b]$ is some finite t-interval, and if y. is a nonnegative random function of half-closed subintervals I_j of I with the properties

(a) $y_{I_1} + y_{I_2} = y_{I_1 I_2}$ for any adjacent subintervals $I_1, I_2 \subset I$

(b) with probability one $E(y_I | t_o, Z(0)) < \infty$

(ii) With probability one, conditional upon t_o, $Z(0)$, for any $I_j \subset I$,

$$\Pi\{I_j \cap \mathcal{F} S_A\} \leq y_I$$

Then $\psi_A(t) = \phi_A(t)(1-F(t))$ is of bounded variation in (a,b).

For the above lemma, t_o is the time of the first occurrence of some event ε, A is some useful class of sets, $S_A = \{t: Z(t) \varepsilon A\}$, $\mathcal{F} S_A$ denotes the frontier or boundary of S_A and Π denotes the cardinality of a set. The original lemma required that (a) hold for any disjoint intervals $I_1, I_2 \subset I$, but since the additivity property was only used in a binary manner over the disection of I, the property need only hold for adjacent subintervals. Since we are only concerned with showing that the distribution of store content has a limit, we can restrict ourselves to the class of sets $\bigstar$ which consists of sets $A_c = \{x: x \leq c\}$.

In order to prove (B3) for the class $\bigstar$ we must find a set function y. which satisfies (i) and (ii) of Lemma 2 above. As suggested in Smith (1955) we define the random variable $\delta.$ as follows:

For any interval $(\tau_{j-1}, \tau_j] = I_j$

$$\delta_{I_j} = \begin{cases} 0 & \text{if } Z(\tau_{j-1}) \in A \text{ and } Z(\tau_j) \in A \\ & \text{or } Z(\tau_{j-1}) \notin A \text{ and } Z(\tau_j) \notin A \\ 1 & \text{if } Z(\tau_{j-1}) \notin A \text{ and } Z(\tau_j) \in A \\ -1 & \text{if } Z(\tau_{j-1}) \in A \text{ and } Z(\tau_j) \notin A \end{cases}$$

We also define a second set function n_{I_j} as the number of inputs into the system in the interval I_j, and take $y_. = 2\, n_. + \delta_.$.

Additivity is clear for $n_.$. To see this for $\delta_.$ it is somewhat easier to think of $\delta_.$ as a sum of two functions $\alpha_.$ and $\beta_.$ where for $I_j = (\tau_{j-1}, \tau_j]$

$$\alpha_{I_j} = \begin{cases} -1/2 & \text{if } \tau_{j-1} \in A \\ 1/2 & \text{if } \tau_{j-1} \notin A \end{cases}$$

$$\beta_{I_j} = \begin{cases} 1/2 & \text{if } \tau_j \in A \\ -1/2 & \text{if } \tau_j \notin A \end{cases}$$

Clearly $\delta_{I_j} = \alpha_{I_j} + \beta_{I_j}$. For additivity we examine

$$\begin{aligned} \delta_{I_j} + \delta_{I_{j+1}} &= (\alpha_{I_j} + \beta_{I_j}) + (\alpha_{I_{j+1}} + \beta_{I_{j+1}}) \\ &= \alpha_{I_j} + (\beta_{I_j} + \alpha_{I_{j+1}}) + \beta_{I_{j+1}} \\ &= \alpha_{I_j} + \beta_{I_{j+1}} \end{aligned}$$

since $\beta_{I_j} = -\alpha_{I_{j+1}}$. But

$$\begin{aligned} \alpha_{I_j} + \beta_{I_{j+1}} &= \alpha_{I_j \cup I_{j+1}} + \beta_{I_j \cup I_{j+1}} \\ &= \delta_{I_j \cup I_{j+1}} \end{aligned}$$

So we have $\delta_{I_j} + \delta_{I_{j+1}} = \delta_{I_j \cup I_{j+1}}$

That (b) is true is obvious, since $E_{y.} \leq 2\, E\, n. + 1$ and $E(n.) < \infty$ since we are dealing with Compound Poisson input.

To show that (ii) is satisfied we must determine how many boundary points there are in S_A which are also in I_j. Let $A_c = [0,c]$. Then $Z(t) \varepsilon A_c$ if and only if $Z(t) \leq c$. Note that if $Z(t) \varepsilon A_c$ then $Z(t+s) \varepsilon A_c$ if there are no inputs to the system in $(t,t+s]$. If there is an input there are two possibilities

(1) The process remains in A_c (hence creates no boundary points)

(2) The process jumps above c. Then we have a boundary point, and possibly another if the process reenters A_c.

Thus an input during $I_j = (\tau_{j-1},\tau_j]$ results in creating at most two boundary points of S_A -- the first if the input is large enough for Z(t) to jump above c, followed by a second if the process falls back below c before τ_j. Let $J_k = (t_{k-1}, t_k]$ where $t_1, t_2, \ldots , t_N$ are times where inputs occur in I_j, $t_o = \tau_{j-1}$, $t_{N+1} = \tau_j$ and $N = n_{I_i}$. (Note, if $t_N = \tau_j$ then we say $J_{N+1} = \phi$.) We can write $I_j = \bigcup_{k=1}^{N+1} J_k$. Certainly $\mathcal{N}(I_j \cap \mathcal{F} S_A) = \sum_{k=1}^{N+1} \mathcal{N}(J_k \cap \mathcal{F} S_{A_c})$. We have

$$\mathcal{N}(J_1 \cap \mathcal{F} S_{A_c}) \leq \begin{cases} 1 & \text{if } Z(\tau_{j-1})\ \varepsilon A_c \\ 2 & \text{if } Z(\tau_{j-1}) \notin A_{c\ j} \end{cases}$$

$$\mathcal{N}(J_k \cap \mathcal{F} S_{A_c}) \leq 2 \qquad k = 2, \ldots , N$$

$$\mathcal{N}(J_{N+1} \cap \mathcal{F} S_{A_c}) \leq \begin{cases} 1 & \text{if } Z(\tau_j)\ \varepsilon A_c \\ 0 & \text{if } Z(\tau_j) \notin A_c \end{cases}$$

Thus

$$n(I_j \cap \mathcal{F} S_A) \le \begin{cases} 1+2(N-1)+1 & \text{if} \quad Z(\tau_{j-1}) \in A_c \text{ and } Z(\tau_j) \in A_c \\ 2+2(N-1)+0 & \text{if} \quad Z(\tau_{j-1}) \notin A_c \text{ and } Z(\tau_j) \notin A_c \\ 2+2(N-1)+1 & \text{if} \quad Z(\tau_{j-1}) \notin A_c \text{ and } Z(\tau_j) \in A_c \\ 1+2(N-1)+0 & \text{if} \quad Z(\tau_{j-1}) \in A_c \text{ and } Z(\tau_j) \notin A_c \end{cases}$$

$$= 2n_{I_j} + \delta_{I_j}$$

We have shown that $y_. = 2\,n_. + \delta_.$ satisfies the assumptions of Lemma 2 of Smith (1955), which implies that $\phi_A(t)\,\{1-F(t)\}$ is of bounded variation for sets A of the type [0,c]. This is sufficient to complete the proof of our theorem, since with (B1)-(B3) satisfied, theorem 2 of Smith (1955) states that the limiting distribution of Z(t) exists. In fact, the theorem specifies that

$$\lim_{t\to\infty} P\,\{Z(t) \le c\} = \frac{1}{\mu} \int_0^\infty \phi_A\,(v)\,(1-F(v))\,d\,v$$

where A = [0,c].

Sometime after this theorem was completed the paper by Brockwell (1977) appeared. His limit theorem is similar to Theorem 2.5, but Brockwell assumed from the outset that r(x) was nondecreasing and continuous, while this theorem does not.

CHAPTER 3

PERTURBATION APPROACH TO THE STORE WITH NONHOMOGENEOUS INPUT

3.1 A Probabilistic Method

In this chapter we present an approach which will allow us to develop formulae for various functions of a storage system with time dependent input. The method is direct and probabilistic in nature, making use of the structure of the input process.

We will be interested, from this time forward, in a process with general release rule, where the number of inputs into the system in any interval (0,t] is a nonhomogeneous Poisson random variable, i.e.

$$P\{1 \text{ input in } (t,t+dt)\} = \lambda(t)dt + o(dt)$$

$$P\{\text{more than 1 input in } (t,t+dt)\} = o(dt)$$

Let N(t) be the number of inputs in (0,t]. Then we have

$$(1) \qquad P\{N(t) = n\} = \frac{[\Lambda(t)]^n}{n!} e^{-\Lambda(t)} \quad ; \quad \Lambda(t) = \int_0^t \lambda(u)du$$

Note that if $\lambda(t) \equiv \lambda$, then $\Lambda(t) = \lambda t$ and (1) is just the familiar homogeneous Poisson probability. If we let B(x) be the distribution function of input size, then the total input in (0,t], A(t), has nonhomogeneous Compound Poisson distribution, with

$$P\{A(t) \leq x\} = \sum_{n=0}^{\infty} e^{-\Lambda(t)} \frac{[\Lambda(t)]^n}{n!} B^{*n}(x)$$

where B^{*n} is the n-fold convolution of B with itself.

The key to the development of our approach will be to assume that the intensity $\lambda(t) = \lambda_o(1+\varepsilon\phi(t))$ where $\lambda_o>0, \varepsilon>0$ and $\phi(t)$ is some measurable nonnegative function. The resulting input process can thus be regarded as the sum of two independent Compound Poisson processes: one with intensity $\lambda_1(t)\equiv\lambda_o$; the other with intensity $\lambda_2(t) = \lambda_o\varepsilon\phi(t)$. We will call the inputs into the system originating from these two sources Type I and Type II inputs respectively. Notice that the Type I input is time homogeneous. This will play an instrumental role in the development of formulae, since, conditional on knowing when the Type II input occur, the storage process has homogeneous input.

In what follows, we will focus our attention on the probability of emptiness, $P\{Z(t)=0\}$. The method developed here can be applied directly to other quantities, such as the mean store content and Laplace-Stieltjes Transform of store content. Let $N_j(t)$ be the number of inputs of Type j which arrive at the store in $(0,t]$, $j=1,2$. Then the probability the store is empty can be written

$$P\{Z(t)=0\} = \sum_{k=0}^{\infty} P\{Z(t)=0 \text{ and } N_2(t)=k\}.$$

We will examine this series term by term. To begin, we note that

$$P\{Z(t)=0 \text{ and } N_2(t)=0\} = P\{Z(t)=0 \mid N_2(t)=0\}\, P\{N_2(t)=0\}$$

If we know there have been no Type II inputs up to time t then the process (up to that time) is equivalent to the homogeneous store with intensity λ_o. Let $\Pi(t,\lambda(\cdot),W)$ denote the probability the store is empty at t, for a store with Compound Poisson input, intensity $\lambda(\cdot)$, and initial content distribution W. Then we have

$$P\{Z(t)=0 \mid N_2(t)=0\} = \Pi(t,\lambda_o,W_o)$$

and thus

$$P\{Z(t)=0 \text{ and } N_2(t)=k\} = \pi(t,\lambda_o,W_o)e^{-\varepsilon\lambda_o\phi(t)}$$

where $\phi(t)=\int_0^t \phi(u)du$ and $W_o(x)$ is the distribution function of Z(0). Next, suppose $N_2(t)=1$. The Type II input will occur at some point u in the interval (0,t]. Let T_i be the time of the i^{th} Type II input. Then

$$P\{Z(t)=0 \text{ and } N_2(t)=1\} = \int_0^t P\{Z(t)=0 \text{ and } N_2(t)=1 \text{ and } T_1\varepsilon du_1\}$$

which is just

$$\int_0^t P\{Z(t)=0 \mid N_2(t)=1 \text{ and } T_1=u_1\}\ P\{N_2(t)=1 \text{ and } T_1\varepsilon du_1\}.$$

The probability of a Type II input in du_1 is $\varepsilon\lambda_o\phi(u_1)du_1 + o(du_1)$ and the probability that this is the only Type II input in (0,t] is

$$e^{-\int_0^{u_1}\varepsilon\lambda_o\phi(x)dx}\ \varepsilon\lambda_o\phi(u_1)du_1 e^{-\int_{u_1}^t \varepsilon\lambda o\phi(x)dx} + o(du_1)=\varepsilon\lambda_o\phi(u_1)e^{-\int_0^t \varepsilon\lambda_o\phi(x)dx}du_1 + o(du_1)$$

We need to determine the probability the store is empty at t, if we know that the first Type II input occurs in du_1 and no more Type II inputs occur before t. The input up to time u_1- is all homogeneous (Type I) input, so at u_1- the distribution of store content is $\overline{W}(x,u_1-,\lambda_o,W_o)$, where $\overline{W}(x,t,\lambda,W)$ denotes the distribution at time t of the content of a store with Compound Poisson input, intensity λ, and initial distribution W(x). At time u_1+ there is an input to the store, and the size of the input is independent of the content at u_1. Thus the distribution of content at u_1+ is the convolution $\overline{W}(x,u,\lambda_o,W_o)*B(x)$. Let $W(x,t,\lambda|u)$ be the content at time t of a homogeneous store which has Compound Poisson input with intensity λ, conditional on knowing that a Type II input occurs in du. Then we can write $W(x,u_1,\lambda_o|u_1) = \overline{W}(x,u_1,\lambda_o,W)*B(x)$.

We use the shorthand notation $W_1(u_1)$ for this distribution function at time u_1.

Since there are no more Type II inputs in $(u_1,t]$, the probability the store will be empty at t is just the probability that a store with homogeneous Compound Poisson input, intensity λ_0, and initial distribution $W(x,u_1,\lambda_0|u_1)$ will be empty $t-u_1$ time units later, or $\Pi(t-u_1,\lambda_0,W_1(u_1))$. We are therefore led to the equation

$$P\{Z(t)=0 \text{ and } N_2(t)=1\} = e^{-\varepsilon\lambda_0\Phi(t)t}\int_0^t \phi(u_1)\Pi(t-u_1,\lambda_0,W_1(u_1))du_1.$$

In a similar fashion, we can see that for $N_2(t)=k$, if the k Type II inputs occur in $du_1, du_2, \ldots, du_k$ then

$$P\{Z(t)=0 \text{ and } N_2(t)=k\} =$$

$$\int_0^t\int_0^{u_k}\int_0^{u_2} e^{-\varepsilon\lambda_0\int_0^{u_1}\phi(x)dx}\,\varepsilon\lambda_0\phi(u_1)\ldots e^{-\varepsilon\lambda_0\int_{u_{k-1}}^{u_k}\phi(x)dx}\,\varepsilon\lambda_0\phi(u_k)\; e^{-\varepsilon\lambda_0\int_{u_k}^{t}\phi(x)dk}\,\Pi(t-u_k,\lambda_0,W_k(u_k))du_1\ldots du_k$$

$$= e^{-\varepsilon\lambda_0\Phi(t)}\varepsilon^k\lambda_0{}^k\int_0^t\int_0^{u_k}\int_0^{u_2}\phi(u_1)\ldots\phi(u_k)\Pi(t-u_k,\lambda_0,W_k(u_k))du_1\ldots du_{k_1}$$

The notation $W_k(u_k)$ represents the distribution function $W(x,u_k,\lambda_0|u_1, \ldots ,u_k)$ where $W(x,t,\lambda|u_1, \ldots u_n)$ is the content at time t of a store which has homogeneous Compound Poisson input with intensity λ_0, conditional on knowing that Type II inputs occur in $du_1, du_2, \ldots,$ du_n. We assume that $W(x,0,\lambda_0|u_1, \ldots ,u_n) = W_0(x)$ for all n.

We have thus derived the following formula for the probability a store will be empty at t:

$$\Pi(t,\lambda(\cdot),W_o)=e^{-\varepsilon\lambda_o\Phi(t)}\{\Pi(t,\lambda_o,W_o)+\varepsilon\lambda_o\int_o^t \phi(u_1)\Pi(t-u_1,\lambda_o,W_1(u_1))du_1$$

(2)

$$+\sum_{k=2}^{\infty}\varepsilon^k\lambda_o^k\int_o^t\int_o^{u_k}\dots\int_o^{u_2}\phi(u_1)\dots\phi(u_k)\Pi(t-u_k,\lambda_o,W_k(u_k))du_1\dots du_k\}$$

An alternate form of (2) expresses $\Pi(t,\lambda(\cdot),W_o)$ as a power series in ε. To do this, simply note that

$$e^{-\varepsilon\lambda_o\Phi(t)} = \sum_{j=o}^{\infty}(-1)^j\frac{[\varepsilon\lambda_o\Phi(t)]^j}{j!}$$

Then if we let $I_k(t)=\int_o^t\int_o^{u_k}\int_o^{u_2}\phi(u_1)\dots\phi(u_k)\Pi(t-u_k,\lambda_o,W_k(u_k))du_1\dots du_k$,

$k>1$, $I_o(t)=\Pi(t,\lambda_o,W_o)$ and $I_1(t)=\int_o^t \phi(u_1)\Pi(t-u_1,\lambda_o,W_1(u_1))du_1$, we can rewrite (2) as

$$(3)\quad \Pi(t,\lambda(\cdot),W_o) = \sum_{j=o}^{\infty}\Pi_j(t)\varepsilon^j$$

where

$$\Pi_j(t) = \lambda_o^j\sum_{k=o}^{j}(-1)^k\frac{\Phi^k(t)}{k!}I(t)_{j-k}$$

Similar equations to (2) and (3) can be written for such quantities as the distribution of store content and mean content.

3.2 Extending the Approach to General $\phi(t)$

All the work so far in this chapter has required that $\phi(t)$ be a non-negative function, since $\varepsilon\lambda_o\phi(t)$ was taken to be the intensity of a

Compound Poisson process. But this assumption is restrictive and undesirable in general. Unfortunately we have been unable to prove a general theorem which applies to a wide range of processes. We feel that, at the very least, the following can be shown to hold.

<u>Conjecture 3.1</u>: For a storage system with nonhomogeneous Compound Poisson input which has intensity $\lambda(t)=\lambda_o(1+\varepsilon\phi(t))$ and a release rule $r(x)$ which is a finite degree polynomial in x, the Laplace-Stieltjes Transform of store content can be written

$$M(s,t,\lambda(\cdot),W_o)=e^{-\varepsilon\lambda_o\Phi(t)}\{M(s,t,\lambda_o,W_o)+\varepsilon\lambda_o\int_o^t \phi(u_1)M(s,t-u_1,\lambda_o,(u_1)\,du$$

(4)

$$+\sum_{k=2}^{\infty}\varepsilon^k\lambda_o^k\int_o^t\int_o^{u_k}\cdots\int_o^{u_2}\phi(u_1)\ldots\phi(u_k)M(s,t-u_k,\lambda_o,W_k(u_k))du_1\ldots du_k\}$$

where $M(s,t,\lambda(\cdot),W)$ is the Laplace-Stieltjes Transform of the content $Z(t)$ of the store with intensity $\lambda(\cdot)$ and initial distribution W.

<u>Outline of Proof</u>: We can show that $M(s,t,\lambda(\cdot),W_o)$ satisfies the following partial differential equation:

$$\frac{\partial M(s,t,\lambda(\cdot),W)}{\partial t}+\lambda(t)(1-B^*(s))\,M(s,t,\lambda(\cdot),W)=$$

(5)

$$s\sum_{k=o}^{m}(-1)^k a_k\frac{\partial^k M(s,t,\lambda(\cdot),W)}{\partial s^k}-a_o s\Pi(t,\lambda(\cdot),W)$$

where $r(x)=\sum_{k=o}^{m}a_k x^k$. We do this in the following Lemma, which we will find useful later also.

<u>Lemma 3.1.1</u>: For a storage system with nonhomogeneous Compound Poisson input which has intensity $\lambda(t)$ and release rule $r(x)$, $M(s,t,\lambda(\cdot),W)$ satisfies the equation

$$(6) \quad \frac{\partial M(s,t,\lambda(\cdot),W)}{\partial t} + \lambda(t)(1-B^*(s))M(s,t,\lambda(\cdot),W) = sE[r(z(t))e^{-sZ(t)}].$$

In addition, if $r(x) = \sum_{k=o}^{m} a_k x^k$ then (6) becomes

$$\frac{\partial M(s,t,\lambda(\cdot),W)}{\partial t} + \lambda(t)(1-B^*(s))M(s,t,\lambda(\cdot),W)$$

$$= s \sum_{k=o}^{m} (-1)^m a_k \partial^k \frac{M(s,t,\lambda(\cdot),W)}{\partial s^k} - a_o s\Pi(t,\lambda(\cdot),W).$$

<u>Proof</u>: We need to consider

$$\frac{\partial M(s,t,\lambda(\cdot),W)}{\partial t} = \lim_{\tau \to o} \frac{M(s,t+\tau,\lambda(\cdot),W) - M(s,t,\lambda(\cdot),W)}{\tau}$$

Let $A_o = A_o(t,\tau) = \{\omega : N(t,\tau,\omega) = 0\}$

$A_1 = A_1(t,\tau) = \{\omega : N(t,\tau,\omega) = 1\}$

$A_2 = A_2(t,\tau) = \{\omega : N(t,\tau,\omega) > 1\}$

where $N(t,\tau,\omega)$ is the number of inputs to arrive in $(t, t+\tau]$. Dropping the parameters λ and W from the notation for the present, we write $M(s,t+\tau)$ as

$$M(s,t+\tau) = \int_\Omega e^{-sZ(t+\tau)} dP$$

$$= \int_{A_o} e^{-sZ(t+\tau)}dP + \int_{A_1} e^{-sZ(t+\tau)}dP + \int_{A_2} e^{-sZ(t+\tau)}dP$$

$$= I_o + I_1 + I_2, \text{ say.}$$

We will examine each integral separately. Note that for small τ, the output of the system in $(t,t+\tau]$ is $r(Z(t))]$. If there is no input in $(t,t+\tau]$ then the content at time $t+\tau$ can be written $Z(t+\tau) = Z(t) - r(Z(t))\tau$ if τ is sufficiently small. For such τ

$$I_o = \int_{A_o} e^{-sZ(t)+s\ r(Z(t))\tau}dP$$

$$= \int_{A_o} e^{-sZ(t)}dP + \int_{A_o} s\tau r(Z(t))\ e^{-sZ(t)}dP + o(\tau)$$

(8) $$= \int_{\Omega} \chi_{A_o} e^{-sZ(t)}dP + \int_{\Omega} \chi_{A_o}\ s\tau r(Z(t))\ e^{-sZ(t)}dP + o(\tau)$$

where χ_A is the indicator function of the set A. Since $A_o(t,\tau)$ is independent of $Z(t)$, (8) can be written

(9) $$I_o = P\{A_o(t,\tau)\}M(s,t)+s\tau P\{A_o(t,\tau)\}E\{r(Z(t))e^{-sZ(t)}\}+o(\tau)$$

For a process with nonhomogeneous Compound Poisson input, if τ is small, $P\{A_o(t,\tau)\} = (1-\lambda(t)\tau) + o(\tau)$. Thus we have

(10) $$I_o = M(s,t)-\lambda(t)\tau M(s,t)+s\tau E\{r(Z(t))e^{-sZ(t)}\}+o(\tau)$$

Suppose there is an input in $(t,t+\tau]$ of size U. Then the content at time $t+\tau$ can be written, for small τ, as $Z(t+\tau)=Z(t)+U\ r(Z(t))\tau$. We use this to write I_1 as

$$I_1 = \int_{A_1} e^{-sZ(t)-su+s\tau r(Z(t))} dP$$

(11)

$$= \int_{A_1} e^{-sZ(t)-su} dP + s\tau \int_{A_1} r(Z(t))\, e^{-sZ(t)-su} dP + o(\tau)$$

Since $A_1(t,\tau)$, U and $Z(t)$ are mutually independent, (11) becomes

$$I_1 = P\{A_1(t,\tau)\}[B^*(s)M(s,t)+B^*(s)s\tau \int_\Omega r(Z(t))e^{-sZ(t)}dP] + o(\tau)$$

Recall that $P\{A_1(t,\tau)\} = \lambda(t)\tau + o(\tau)$. Thus

(12) $I_1 = \lambda(t)\tau B^*(s)M(s,t) + o(\tau)$

The third integral, I_2, is nonnegative and bounded above by a function of order $o(\tau)$. We see this as follows:

(13) $I_2 = \int_{A_2} e^{-sZ(t+\tau)}dP \leq \int_{A_2} 1dP = P\{A_2(t,\tau)\} = o(\tau)$.

We can combine (10), (12), and (13) to show that

$$\frac{\partial M(s,t)}{\partial t} = -\lambda(t)M(s,t)+s\, E\{r(Z(t))e^{-sZ(t)}\} + \lambda(t)B^*(s)M(s,t)$$

Suppose $\tilde{r}(x) = \sum_{k=o}^{\infty} a_k x^k$, and let $r(x) = \tilde{r}(x)$ for $x>o$. Then

$$E\{r(Z(t))e^{-sZ(t)}\} = \int_\Omega r(Z(t))e^{-sZ(t)}dP$$

$$= \int_{\{Z(t)>o\}} r(Z(t))e^{-sZ(t)}dP$$

$$= \int_{\Omega} \tilde{r}\,(Z(t))e^{-sZ(t)}dP - \tilde{r}(0)\Pi(t)$$

$$= \sum_{k=o}^{\infty} a_k \int_{\Omega} Z^k(t)e^{-sZ(t)}dP - a_o\Pi(t)$$

(14) $$= \sum_{k=o}^{m} a_k(-1)^k \frac{\partial M^k(s,t)}{\partial s^k} - a_o\Pi(t)$$

Substituting (14) into (5) gives the desired conclusion.

Outline of Proof continued: We also know that (4) holds for $\phi(t)>o$. This can be shown by using the perturbation argument presented earlier in this chapter. Thus $M(s,t,\lambda(\cdot),W)$ as defined in (4) must satisfy (5). It seems clear that (4) will formally satisfy the partial differential equation, regardless of the range of values assumed by $\phi(t)$. If we can show that (5) has a unique solution which satisfies the boundary conditions

(15)
$$M(0,t,\lambda(\cdot),W) = 1$$

$$M(s,0,\lambda(\cdot),W) = \int_0^{\infty} e^{-sx}dW(x)$$

then we will have shown that (4) is the Laplace-Stieltjes Transform we seek.

For the store with $r(x)\equiv 1, x>0$, Reich (1958) has shown that there is a unique probabilistic solution to (5). For higher order polynomials, we have been unable to find appropriate criteria for the existence of a unique solution.

Corollary 3.1.1: For a store as described in Conjecture 3.1 the following hold:

$$(16)\quad \Pi(t,\lambda(\cdot),W_o)=e^{-\varepsilon\lambda_o\Phi(t)}\{\Pi(t,\lambda_o,W_o)+\varepsilon\lambda_o\int_o^t \phi(u_1)\Pi(t-u_1,\lambda_o,W_1(u_1))\,du_1$$

$$+\sum_{k=2}^{\infty}\varepsilon^k\lambda_o^k\int_o^t\int_o^{u_k}\cdots\int_o^{u_2}\phi(u_1)\ldots\phi(u_k)\Pi(t-u_k,\lambda_o,W_k(u_k))du_1\ldots du_k\}$$

$$(17)\quad \mu(t,\lambda(\cdot),w_o)=e^{-\varepsilon\lambda_o\Phi(t)}\{\mu(t,\lambda_o,W_o)+\varepsilon\lambda_o\int_o^t \phi(u_1)\mu(t-u_1,\lambda_o,W_1(u_1))du_1$$

$$+\sum_{k=2}^{\infty}\varepsilon^k\lambda_o^k\int_o^t\int_o^{u_k}\cdots\int_o^{u_2}\phi(u_1)\ldots\phi(u_k)\mu(t-u_k,\lambda_o,W_k(u_k))du_1\ldots du_k\}$$

where μ is the expected store content.

<u>Proof</u>: Recall that $\lim_{s\to\infty} M(s,t,\lambda(\cdot),W_o)=\Pi(t,\lambda(\cdot),W_o)$. Taking the limit as s goes to infinity in (4) yields (16). Note also that $\lim_{s\to 0} \partial M(s,t,\lambda(\cdot),W_o)/\partial s=\mu(t,\lambda(\cdot),W_o)$. Hence using (4) we can get (17).

Some comments are in order about our chief assumption--that the intensity of our input process can be written in the form $\lambda(t)=\lambda_o(1+\varepsilon\phi(t))$. At first glance this might be thought to be rather restrictive. Actually, it is not. Suppose we have an intensity $\lambda(t)>0$. For any choice of λ_o and ε we can define $\phi(t)$ as $\phi(t)=[\lambda(t)-\lambda_o]/\varepsilon\lambda_o$.

Of course, more suitable choice of λ_o, ε and ϕ can be made. If $\lambda(t)$ is periodic, we would want λ_o to be the average intensity per period. Then $\lambda_o=\int_o^{\tilde{\omega}}\lambda(u)du$ and $\phi(t)=\dfrac{\lambda(t)-\int_o^{\tilde{\omega}}\lambda(u)du}{\varepsilon\int_o^{\tilde{\omega}}\lambda(u)du}$

where $\tilde{\omega}$ is the period. If we know that $\lambda(t)\to L$ as $t\to\infty$ then a good choice for λ_o would be L.

3.3 An Exact Limiting Result

It is well known that in general, the transient probability $\Pi(t,\lambda_o,W_k)$ is difficult to calculate. Thus determining $\Pi(t,\lambda(\cdot),W_o)$ will be even more difficult, as it involves an integral of which $\Pi(t,\lambda_o,W_k(u_k))$ is just a part. For this reason, we will concentrate on results concerning the limiting value of Π.

Suppose we know that $\lambda(t)$ converges to a finite limit. In particular, let $\lambda(t)=\lambda_o(1+\varepsilon\phi(t))\to\lambda_o$. We might expect intuitively that $\Pi(t,\lambda(\cdot),W_o)\to\Pi(\infty.\lambda_o)$. With only some minor assumptions on $\phi(t)$ we can show that this is indeed the case. Theorem 3.2, which proves this, is essentially an extension of part of Theorem 3 of Takács (1955) for the M/G/1 queue.

Before proving the theorem, we will first prove a few lemmas which will be useful both here and elsewhere. The first shows that if the intensity of the input process is bounded, then the probability of emptiness and the content moments are bounded.

<u>Lemma 3.2.1</u>: Suppose we have a storage system which has nonhomogeneous Compound Poisson input with intensity $\lambda(t)$ where $M_1(t)\leq\lambda(t)\leq M_2(t)$ for some positive functions M_1 and M_2. Let $Z_i(t)=Z(t,\omega_i,\xi,r)$ denote the store content with intensity $M_i(t)$, i=1,2 and $Z(t)=Z(t,\omega,\xi,r)$ denote the content with intensity $\lambda(t)$. Then

$$P\{Z_2(t)=0\}\leq P\{Z(t)=0\}\leq P\{Z_1(t)=0\}$$

$$E\{Z_1^k(t)\} \leq E\{Z^k(t)\} \leq E\{Z_2^k(t)\}$$

<u>Proof</u>: Suppose we have a process $Z_3(t)=Z(t,\omega_3,\xi,r)$ where the Z_3 - process has two independent input sources, both with Compound Poisson input and intensities $\lambda_1(t)=\lambda(t)-M_1(t)$ and $\lambda_2(t)=M_1(t)$. Let $N_i(t)$ be the number of inputs in $(0,t]$ from the source with intensity $\lambda_i(t)$, i=1,2. If, for the Z_3 - process, we know that $N_1(t)=0$ then

$$P\{Z_3(t) = 0 \mid N_1(t) = 0\} \geq P\{Z_3(t) = 0\}$$

and $$E\{Z_3^k(t) \mid N_1(t) = 0\} \leq E\{Z_3^k(t)\}$$

But if the Z_3 - process only has input from its second source, then since $\lambda_2(t) = M_1(t)$

$$P\{Z_3(t) = 0 \mid N_1(t) = 0\} = P\{Z_1(t) = 0\}$$

and $$E\{Z_3^k(t) \mid N_1(t) = 0\} = E\{Z_1^k(t)\}$$

Note that the overall input rate of the Z_3 - process is $(\lambda(t)-M_1(t))+M_1(t) = \lambda(t)$, which implies that

$$P\{Z_3(t) = 0\} = P\{Z(t) = 0\}$$

and $$E\{Z_3^k(t)\} = E\{Z^k(t)\}$$

Thus we have

$$P\{Z(t) = 0\} \leq P\{Z_1(t) = 0\}$$

and $$E\{Z^k(t)\} \geq E\{Z_1^k(t)\}$$

Similarly we can obtain the other inequalities by introducing a process $Z_4(t)=Z(t,\omega_4,\xi,r)$ where the Z_4 - process has two independent input

sources, both with Compound Poisson input and intensities $\lambda_1(t)=M_2(t)-\lambda(t)$ and $\lambda_2(t)=\lambda(t)$. Then

$$P\{Z(t)=0\}=P\{Z_4(t)=0\}\geq P\{Z_4(t)=0 \mid N_1(t)=0\} = P\{Z_2(t)=0\}$$

$$E\{Z^k(t)\} = E\{Z_4^k(t)\} \leq E\{Z_4^k(t) \mid N_1(t)=0\} = E\{Z_2^k(t)\}.$$

This second lemma is a basic limiting result which will be used often.

Lemma 3.2.2: For any $\psi\varepsilon L_1$

A) If $f(x)$ is a function such that $|f|\leq k$ and $f(x)\to L$ as $x\to\infty$ then

$$\lim_{t\to\infty} \int_0^t \psi(u)f(t-u)du = L \int_0^\infty \psi(u)du$$

B) If $f(x)$ is a periodic function, period $\tilde{\omega}$, and $f(k\tilde{\omega}+\tau)\to f^*(\tau)$ as

$k\to\infty$, $0\leq\tau<\omega$, then $\lim\limits_{k\to\infty} \int_0^{k\tilde{\omega}+\tau} \psi(u)f(k\tilde{\omega}+\tau-u)du = \int_0^\infty \psi(u)f^*(\tau-u)du$

Proof: To prove part (A) we note that since $|f|\leq k$ we have $|\psi(u)f(t-u)| \leq K|\psi(u)|$, and $K\psi(u)\varepsilon L_1$. Also, $\psi(u)f(t-u)\to L\psi(u)$ as $t\to\infty$. Thus by dominated convergence

$$\lim_{t\to\infty} \int_0^t \psi(u)f(t-u)du = L \int_0^\infty (u)du.$$

Part (B) follows in a similar fashion. In this case we note that $\lim\limits_{k\to\infty}\psi(u)f(k\tilde{\omega}+\tau-u)=\psi(u)f(\tau-u)$. Thus by dominated convergence

$$\lim_{k\to\infty} \int_0^{k\tilde{\omega}+\tau} \psi(u)f(k\tilde{\omega}+\tau-u)du = \int_0^\infty \psi(u)f(\tau-u)du.$$

Theorem 3.2: Let $Z(t)$ be the content of a Type A storage process with nonhomogeneous Compound Poisson input which has intensity $\lambda(t)=\lambda_0(1+\varepsilon\phi(t)),\varepsilon>0$ where $\phi(t)\geq 0$, $\phi\varepsilon L_1$ and $\phi(t)\to 0$ as $t\to\infty$. If the

corresponding homogeneous storage process with intensity λ_o has a limiting distribution of store content, then

$$\lim_{t\to\infty} \Pi(t,\lambda(\cdot),W) = \Pi(\infty,\lambda_o)$$

<u>Proof</u>: By Lemma 3.2.1, since $\lambda_o \le \lambda(t)$ we know

$$(18) \qquad \Pi(t,\lambda_o,W_o) \geq \Pi(t,\lambda(\cdot),W_o)$$

Taking upper limits of both sides of (18) yields

$$\Pi(\infty,\lambda_o) \geq \overline{\lim_{t\to\infty}}\, \Pi(t,\lambda(\cdot),(W_o)$$

where $\Pi(\infty,\lambda_o) = \lim_{t\to\infty} \Pi(t,\lambda_o,W_o)$ exists by assumption.

We can use (16) to obtain the other inequality. By Lemma 2.1.1 we have $P\{Z(t)=0|Z(0)=0\} \geq P\{Z(t)=0|Z(0)\geq 0\}$ and hence $\Pi(t-u_k,\lambda_o,W_k) \leq \Pi(t-u_k,\lambda_o,D)$ where $D(x)$ is the degenerate distribution function $D(0)=1$. Then we have the inequality

$$\Pi(t,\lambda(\cdot),W_o) \leq e^{-\varepsilon\lambda_o\Phi(t)}\{\Pi(t,\lambda_o,W_o)+\varepsilon\lambda_o \int_0^t \phi(u)\Pi(t-u,\lambda_o,D)du$$

$$+ \sum_{k=2}^{\infty} \varepsilon^k\lambda_o^k \int_0^t \int_0^{u_k} \cdots \int_0^{u_2} \phi(u_1)\ldots\phi(u_k)\Pi(t-u_k,\lambda_o,D)du_1\ldots du_k\}$$

$$= e^{-\varepsilon\lambda_o\Phi(t)}\{\Pi(t,\lambda_o,W_o)+ \int_0^t \varepsilon\lambda_o\phi(u)\Pi(t-u,\lambda_o,D)du$$

$$+ \sum_{k=2}^{\infty}\varepsilon^k\lambda_o^k \int_0^t \phi(u_k)\Pi(t-u_k,\lambda_o,D) \int_0^{u_k} \cdots \int_0^{u_2} \phi(u_1)\ldots\phi(u_{k-1})du_1\ldots du_k\}$$

The summation can be simplified using the identity

$$(20)\quad \int_0^{x_k}\cdots\int_0^{x_1}\phi(x_o)\ldots\phi(x_{k-1})dx_o\ldots dx_{k-1} = \frac{\left[\int_0^{x_k}\phi(u)du\right]^k}{k!}$$

This identity can be easily proved by induction. The case k=1 is obvious. Assume (20) is true for k=n. For notational convenience, let

$$\Phi_k(x_k) = \int_0^{x_k}\cdots\int_0^{x_1}\phi(x_o)\ldots\phi(x_{k-1})dx_o\ldots dx_{k-1} \quad \text{then}$$

$$\Phi_{n+1}(x_{n+1}) = \int_0^{x_{n+1}}\int_0^{x_n}\cdots\int_0^{x_1}\phi(x_o)\ldots\phi(x_{n-1})dx_o\ldots dx_n$$

$$= \int_0^{x_{n+1}}\phi(x_n)\Phi_n(x_n)dx_n$$

$$= \int_0^{x_{n+1}}\phi(x_n)\frac{\left[\Phi_1(x_n)\right]^n}{n!}dx_n$$

since $\frac{d\Phi_1(x_n)}{dx_n} = \phi(x_n)$ we have

$$\Phi_{n+1}(x_{n+1}) = \int_0^{x_{n+1}}\frac{\Phi_1^n(x_n)}{n!}d\Phi_1(x_n)$$

$$= \frac{\Phi_1^{n+1}(x_{n+1})}{(n+1)!}$$

as claimed.

Using (20) we obtain

$$\Pi(t,\lambda(\cdot),W_o) \geq e^{-\varepsilon\lambda_o\Phi(t)}\{\Pi(t,\lambda_o,W_o)+\varepsilon\lambda_o \int_0^t \phi(u)\Pi(t-u,\lambda_o,D)du$$

$$+\sum_{k=2}^{\infty} \varepsilon^k\lambda_o^k \int_0^t \frac{\phi(u)\Phi^{k-1}(u)}{(k-1)!}\Pi(t-u,\lambda_o,D)du\}.$$

Since $\phi\varepsilon L_1$ we have that

$$\lim_{t\to\infty} \int_0^t \frac{\phi(u)\Phi^{k-1}(u)du}{(k-1)!} = \lim_{t\to\infty} \frac{\left[\int_0^t \phi(u)du\right]^k}{k!} < \infty$$

i.e. $\phi\Phi^{k-1}\varepsilon L_1$. In addition, as $\Pi(t,\lambda_o,W_o)$ is bounded and approaches a limit, we can use Lemma 3.2.2 to show

$$\lim_{t\to\infty} \int_0^t \phi(u)\frac{\Phi^{k-1}(u)}{(k-1)!}\Pi(t-u,\lambda_o,D)du =$$

$$\Pi(\infty,\lambda_o) \int_0^\infty \phi(u)\frac{\Phi^{k-1}(u)du}{(k-1)!} =$$

$$\Pi(\infty,\lambda_o)\frac{\Phi^k(\infty)}{k!}.$$

This yields

$$\underline{\lim}\, \Pi(t,\lambda(\cdot),W_o) \geq e^{-\varepsilon\lambda_o\Phi(\infty)}\{\Pi(\infty,\lambda_o)+\varepsilon\lambda_o\Pi(\infty,\lambda_o)\Phi(\infty)$$

$$+\sum_{k=2}^{\infty}\varepsilon^k\lambda_o^k\Pi(\infty,\lambda_o)\frac{\Phi^k(\infty)}{k!}\}$$

$$= e^{-\varepsilon\lambda_o\Phi(\infty)}\Pi(\infty,\lambda_o)e^{\varepsilon\lambda_o\Phi(\infty)}$$

$$= \Pi(\infty,\lambda_o).$$

From (19) we have $\overline{\lim}_{t\to\infty} \Pi(t, (\cdot),W_o) \le \Pi(\infty,\lambda_o)$ and from (21) we have

$\underline{\lim}_{t\to\infty} \Pi(t,\lambda(\cdot),W_o) \le \Pi(\infty,\lambda_o)$, so we have

$$\Pi(\infty,\lambda_o) \le \underline{\lim}_{t\to\infty} \Pi(t,\lambda(\cdot),W_o) \le \overline{\lim}_{t\to\infty} \Pi(t,\lambda(\cdot),W_o) \le \Pi(\infty,\lambda_o)$$

which gives us our conclusion.

Note that we have not assumed here that $\varepsilon<1$. Thus Theorem 3.2 can apply to any function $\lambda(t)$ which can be written in the form $a+b(t)$ where $a>0, b(t)>0$ and $b\varepsilon L_1$, since we can write $\lambda(t)=a(1+\frac{1}{a}b(t))$. Or, suppose that $\phi(t)\to L$ as $t\to\infty$, $L>0$. If $\phi(t)\ge L$ and $\tilde{\phi}(t)=\phi(t)-L$ then we can write $\lambda(t)=\tilde{\lambda}(1+\tilde{\varepsilon}\phi(t))$ where $\tilde{\lambda}=\lambda_o(1+\varepsilon L)$ and $\tilde{\varepsilon}=\varepsilon/(1+\varepsilon L)$. If in addition $\tilde{\phi}\varepsilon L$, then Theorem 3.2 applies and $\Pi(t,\lambda(\cdot),W_o)\to\Pi(\infty,\lambda_o(1+\varepsilon L))$ as $t\to\infty$.

3.4 Periodic Intensity

The situation in the preceding section is the exception rather than the rule. In most cases we will not be able to get such simple answers. It is impractical to attempt to evaluate all of the terms in the expansion of Π, or in fact many at all. We can perhaps hope to calculate a few terms, leaving a remainder term of order $O(\varepsilon^k)$ as $\varepsilon\to 0$. To find the probability the store is empty at t, up to an error of order of magnitude $O(\varepsilon^k)$ we need to consider at most k-1 inputs of Type II occurring in (0,t]. To see this, recall that

$$P\{Z(t)=0\}=\sum_{n=o}^{\infty} P\{Z(t)=0 \text{ and } N_2(t)=n\}$$

$$=\sum_{n=o}^{k-1} P\{Z(t)=0 \text{ and } N_2(t)=n\}$$

$$+ P\{Z(t)=0 \mid N_2(t)\geq k\}\, P\{N_2(t)\geq k\}$$

and

$$P\{N_2(t)\geq k\} = \sum_{n=k}^{\infty} e^{-\varepsilon\lambda_o\Phi(t)} \frac{[\varepsilon\lambda_o\Phi(t)]^n}{n!} = O(\varepsilon^k).$$

We will only look at the first two terms of the ε-expansion, which will give us the probability the store is empty at t, up to an error of $O(\varepsilon^2)$; i.e. we look at

(22) $\Pi(t,\lambda(\cdot),W_o)=\Pi(t,\lambda_o,W_o)(1-\varepsilon\lambda_o\Phi(t))+\varepsilon\lambda_o \int_0^t \phi(u)\Pi(t-u,\lambda_o,W,(u))du+R_2$

where $R_2=O(\varepsilon^2)$.

This approximation, while appearing simple enough, is not easy to evaluate. The difficulty lies in the fact that W_1 is a function of u. Since we are interested in the limiting distribution of Π, this problem can be alleviated. As we have seen in Chapter 2, under many circumstances a limiting distribution of store content will exist, independent of the initial content distribution. Thus an advantageous choice of initial distribution could simplify things considerably. We will see that very often a good candidate for this is the limiting value of $W(x,t,\lambda_o,W_o)$ which we will denote $\tilde{W}(x,\lambda_o)$. If this is taken as the initial distribution, then $W(x,u_1,\lambda_o|u_1)=\tilde{W}(x,\lambda_o)* B(x)$. Note that this is not dependent

on u_1. For this reason, we will henceforth write $W_1(u_1)=W_1$ in this situation. This will prove helpful, as we will soon see.

In order to be able to take advantage of the limit theorems of Chapter 2, we assume here that we have periodic intensity, with $\phi(t+k)=\phi(t)$ and $\phi(t)$ is Riemann integrable. Then ϕ can be expressed as a Fourier Series

$$(23)\quad \phi(u)=a_o+\sum_{m=-\infty}^{\infty}(b_m\cos 2\pi mu+c_m\sin 2\pi mu)$$

$$=\sum_{m=-\infty}^{\infty}a_m e^{2\pi imu}$$

It will be convenient to assume that $\int_0^1 \phi(u)du=0$; i.e. $a_o=0$. If not, then we can define a new function $\tilde{\phi}(u)=\phi(u)-\int_0^1\phi(x)dx$. In this case $\lambda(u)=\tilde{\lambda}(1+\tilde{\varepsilon}\tilde{\phi}(u))$ where $\tilde{\lambda}=\lambda_o(1+\varepsilon\int_0^1\phi(x)dx)$ and $\tilde{\varepsilon}=\varepsilon/(1+\varepsilon\int_0^1\phi(x)dx)$. It will also simplify matters to assume ϕ can be expressed as a finite series.

We can substitute (23) into (22) to get the approximation

$$(24)\quad \Pi(t,\lambda(\cdot),\tilde{W})=\Pi(t,\lambda_o,\tilde{W})(1-\varepsilon\phi(t))$$

$$+\sum_{m=-M}^{M}\varepsilon\lambda_o\int_0^t a_m e^{2\pi imu}\Pi(t-u,\lambda_o,W_1)du+R_2$$

Making the substitution $\hat{\Pi}(t,\lambda_o,W_1)=\Pi(t,\lambda_o,W_1)-\Pi(\infty,\lambda_o)$ and noting that $\Pi(t,\lambda_o,\tilde{W})=\Pi(\infty,\lambda_o)$ we can rewrite (24) as

$$\Pi(t,\lambda(\cdot),\tilde{W})=\Pi(\infty,\lambda_o)+\sum_{m=-M}^{M}\varepsilon\lambda_o e^{2\pi imt}a_m\int_0^t e^{-2\pi imu}\hat{\Pi}(u,\lambda_o,W_1)du+R_2$$

If the storage process satisfies appropriate conditions, such as those of Theorems 2.2-2.4 then we should be able to find $\Pi(\tau,\lambda(\cdot))=\lim_{n\to\infty}\Pi(n+\tau,\lambda(\cdot),W)$, $0\leq\tau<1$. We need to evaluate terms of the form

$$(25)\qquad \lim_{n\to\infty} e^{2\pi im(n+\tau)}\int_0^{n+\tau} e^{-2\pi imu}\,\hat{\Pi}(u,\lambda_0,W_1)du.$$

We claim that this term equals $e^{2\pi im\tau}\,\hat{\Pi}^0(2\pi im,\lambda_0,W_1)$.

First we note that if $\hat{\Pi}\varepsilon L_1$ then since

$$|\,e^{-\varepsilon\pi imu}\,\hat{\Pi}(u,\lambda_0,W_1)|\leq|\hat{\Pi}(u,\lambda_0,W_1)|,$$

by Dominated Convergence

$$\lim_{n\to\infty}\int_0^{n+\tau} e^{-2\pi imu}\,\hat{\Pi}(u,\lambda_0,W_1)du = \int_0^\infty e^{2\pi imu}\,\hat{\Pi}(u,\lambda_0,W_1)du$$
$$= \hat{\Pi}^0(2\pi im,\lambda_0,W_1).$$

To show that this integral is meaningful, we look at

$$\lim_{n\to\infty}\hat{\Pi}^0(\tfrac{1}{n}+i\beta,\lambda_0,W_1) = \int_0^\infty e^{-(1/n+i\beta)x}\,\hat{\Pi}(x,\lambda_0,W_1)dx.$$

We have $|e^{-(1/n+i\beta)x}\,\hat{\Pi}(x,\lambda_0,W_1)|\leq|\hat{\Pi}(x,\lambda_0,W_1)|$, so by Dominated Convergence again,

$$\lim_{n\to\infty}\hat{\Pi}^0(\tfrac{1}{n}+i\beta,\lambda_0,W_1) = \int_0^\infty \lim_{n\to\infty} e^{-(1/n+i\beta)x}\,\hat{\Pi}(x,\lambda_0,W_1)dx$$

$$= \int_0^\infty e^{-i\beta x} \hat{\Pi}(x,\lambda_0,W_1)dx$$

$$= \hat{\Pi}^0(i\beta)$$

Having determined the limit in (25), we can now write

$$(26) \quad \Pi(\tau,\lambda(\cdot)) = \Pi(\infty,\lambda_0) + \sum_{m=-M}^{M} \varepsilon\lambda_0 a_m \; e^{2\pi i m} \; \hat{\Pi}^0(2\pi i m,\lambda_0,W_1) + R_2$$

Of course, the above limiting formula is contingent on $\hat{\Pi}\varepsilon L_1$. We prove this in the following theorem.

Theorem 3.3: Consider a Type A storage process with Compound Poisson input and release rule $r(x)$ such that there exists a level c where

$$r(x) \geq \alpha > \lambda\beta_1 \qquad \text{for } x > c$$

$$r(x) \leq M_c \qquad \text{for } x \leq c$$

for some constants α and M_c. Let $\hat{\Pi}(t,\lambda,W) = \Pi(t,\lambda,W) - \Pi(\infty,\lambda)$. If the first two moments of the input size distribution (β_1, β_2) are finite, then $\hat{\Pi}\varepsilon L_1$.

Proof: Let ε_c be the event: ε_c occurs at t if $Z(t)=c$ and $Z(t-)\geq c$. Define the following:

$\tilde{q}(t,u)$ = probability the store is empty at t, given ε_c occurred at u, and ε_c has not yet recurred.

$\bar{q}(t)$ = probability the store is empty at t and ε_c has never occurred.

$G(t)$ = distribution of the time to the first occurrence of ε_c

$F(t)$ = distribution of the time between successive occurrences of ε_c

Note that since ε_c is regenerative, $\tilde{q}(t,u)$ is a function of t-u. We will write $\tilde{q}(t,u)=q(t-u)$.

We have the following equation:

$$\Pi(t,\lambda,W) = \bar{q}(t) + \int_0^t q(t-u)dG(u) +$$

$$\int_0^t q(t-u)d[G*F(u)] + \int_0^t q(t-u)d[G*F*F(u)] + \ldots$$

$$= \bar{q}(t) + \int_0^t q(t-u)d\,H(u)$$

where $H(u)=G(u) + G*F(u) + G*F*F(u) + \ldots.$ Notice that $H(u)$ satisfies the renewal equation $H(u)=G(u) + \int_0^u H(u-x)dF(x)$ and thus is a renewal function.

We need to determine $\Pi(\infty,\lambda_0)$. We have shown previously (in the proof of Theorem 2.5) that ε_c is certain, so $\bar{q}(t)\leq P\{\varepsilon_c$ has not occurred by $t\}\to 0$ as $t\to\infty$. If $q(t)$ can be shown to be of bounded variation on every finite interval, and $q\varepsilon L_1$, then by the Key Renewal Theorem we will have

$$\lim_{t\to\infty} \int_0^t q(t-u)dH(u) = \frac{1}{\mu_1} \int_0^\infty q(u)du$$

where $\mu_1 = \int_0^\infty udF(u)$. We have $q(u)\leq \int_u^\infty dF(u) = 1-F(u)$ and

$\int_0^\infty q(u)du \leq \int_0^\infty (1-F(u))du = \mu_1<\infty$, by Lemma 2.5.1. Thus $q\varepsilon L_1$.

To prove bounded variation we refer back to the proof of Theorem 2.5. There we saw that for the class $\bigstar$ of intervals [0,a], we can use Lemma

2 of Smith (1955) to show that the function $\psi_A(t)=\phi_A(t)(1-F(t))$ is of bounded variation in every finite interval. Let $A=\{0\}$. We have $A\varepsilon\mathcal{A}$. Note that if we suppose ε_c occurs at t_o and its next occurrence is at t_o+t_1 then

$$\phi_A(t) = P\{Z(t_o+t) = 0 \mid Z(0),t_o,t_1>t\} = q(t)$$

is the function we are interested in.

The function $1-F(t)$ is monotone, so $(1-F(t))^{-1}$ is also monotone. Then $(1-F(t))^{-1}$ is of bounded variation on every bounded interval. This implies that $\psi_A(t)/(1-F(t))$ is also of bounded variation, since the product of two functions of bounded variation is again a function of bounded variation. But $\psi_A(t)/(1-F(t))=\phi_A(t)=q(t)$. Thus we have shown that the conditions of the Key Renewal Theorem are met.

With this we have that

$$\text{(27)}\quad \hat{\Pi}(t,\lambda_o,W) = \bar{q}(t) + \int_0^t q(t-u)dH(u) - \frac{1}{\mu_1}\int_0^\infty q(u)du$$

$$= \bar{q}(t) + \int_0^t q(t-u)d(H(x) - \frac{1}{\mu_1}) - \frac{1}{\mu_1}\int_t^\infty q(u)du.$$

Now $\bar{q}(u) \leq \int_u^\infty dG(v) = 1-G(u)$, so $\int_0^\infty \bar{q}(u)du = \int_0^\infty (1-G(u))du = \nu_1$, say, but $\nu_1 \leq \mu_1<\infty$; i.e. $\bar{q}\varepsilon L_1$. Looking at the third term of (27) we have

$$\int_0^\infty \int_t^\infty q(u)dudt \leq \int_0^\infty \int_t^\infty (1-F(u))dudt$$

$$= \int_0^\infty \int_0^u (1-F(u))dtdu$$

$$= \int_0^\infty u(1-F(u)du$$

$$= \frac{1}{2} \int_0^\infty u^2 dF(u)$$

$$= \frac{1}{2} \mu_2 < \infty$$

by Lemma 2.5.1.

For the second term in (27) to be in L_1, we must have

$$(28) \quad \int_0^\infty \left| \int_0^t q(t-x)d(H(s) - \frac{x}{\mu_1}) \right| dt < \infty$$

We have the inequality

$$\int_0^\infty \left| \int_0^t q(t-x)d(H(x) - \frac{x}{\mu_1}) \right| dt \leq$$

$$\int_0^\infty \int_0^t q(t-x) \left| d(H(x) - \frac{x}{\mu_1}) \right| dt.$$

By Fubinis Theorem this can be written

$$\int_0^\infty \int_x^\infty q(t-x)dt \left| d(H(x) - \frac{x}{\mu_1}) \right|.$$

The inner integral is $\int_0^\infty q(t-u)dt = \int_0^\infty q(v)dv < \infty$ since $q \epsilon L_1$.

We know that H(x) is a renewal function of a delayed renewal process, where the distribution function of the time until the initial event is

$G(x)$. Let $H_o(x)$ be the renewal function of the ordinary renewal process. Then $H(x)=G(x)+G*H_o(x)$. Let $D(x)=H_o(x) - \frac{x}{\mu_1}$ we have

$$G * D(x) = \left[G * H_o(x) - \frac{x}{\mu_1} \right] + \left[\frac{x}{\mu_1} - \frac{1}{\mu_1} \int_0^x G(y)dy \right]$$

By Theorem 8 of Smith (1954), if $\mu_1 < \infty$ and $\mu_2 < \infty$, then

$\int_0^\infty |dH_o(x) - \frac{x}{\mu_1}| < \infty$ as long as $F(x)$ has an absolutely continuous component. We would like to write $F(x)=L(x)+A(x)$ where $A(x)$ is an absolutely continuous function. Let X be the time between successive occurrences of ε_c, and let $A(x)=P\{X \leq x$ and no input in $(0,t^*]\}$ where t^* is the number which makes the following hold:

$$t^* = \int_0^c \frac{dx}{r(x)} .$$

Thus t^* is the time the store will first empty if we start at level c and there are no inputs in $(0,t^*]$.

We have $A(x)=0$, $x \leq t^*$. Let $K_c(x)$ be the distribution function of the time from the first input to an empty store until the content next crosses c from above. Then for $x > t^*$

$$A(x) = e^{-\lambda t^*} \int_0^{x-t^*} \lambda e^{-\lambda u} K_c(x-t^*-u)du.$$

Let $A_1(x) = 1-e^{-\lambda x}$ and $A_2(x) = e^{-\lambda t^*} K_c(x)$. Then $A(x) = A_1 * A_2(x-t^*)$.

Since A_1 is absolutely continuous, so is A. Thus $\int_0^\infty | dD(x) | < \infty$.

Since $G(x) \leq 1$ $\int | dG * D(x) | \leq \int | dD(x) | < \infty$. If we can show

$$\int_0^\infty | d(\frac{x}{\mu_1} - \frac{1}{\mu_1} \int_0^x G(y)dy) | < \infty$$

then we will have $\int_0^\infty | d(G * H_o(x) - \frac{x}{\mu_1})| < \infty$. But

$$\int_0^\infty | d(\frac{x}{\mu_1} - \frac{1}{\mu_1} \int_0^x G(y)dy) | = \int_0^\infty | \frac{1}{\mu_1} - \frac{G(x)}{\mu_1} | \, dx$$

$$= \frac{1}{\mu_1} \int_0^\infty (1-G(x))dx < \infty$$

since

$$\int | d(H(x) - \frac{x}{\mu_1})| \leq \int |dG(x)| + \int | d(G * H_o(x) - \frac{x}{\mu_1})|$$

we thus have $\int_0^\infty | d(H(x) - \frac{x}{\mu_1})| < \infty$. All this implies that (28) holds,

so $\int_0^t q(t-x)d(H(x) - \frac{x}{\mu_1}) \in L_1$. We therefore have shown that $\hat{\Pi}(x,\lambda,W)\varepsilon L_1$.

3.5 An Example: $\phi(x)=\cos 2\pi x$

The formula in (26) is not as formidable as it seems. We look at a simple case. Suppose $\phi(t)=\cos 2\pi t$. This can be written

$\phi(t) = \frac{1}{2} e^{-2\pi it} + \frac{1}{2} e^{2\pi it}$. This makes (26) look like

$$(29) \quad \Pi(\tau,\lambda(\cdot))=\pi(\infty,\lambda_o)+\varepsilon\lambda_o\Big[\frac{1}{2} e^{-2\pi i\tau} \hat{\Pi}^o(-2\pi i,\lambda_o W_1)+\frac{1}{2} e^{2\pi i\tau} \hat{\Pi}^o(2\pi i,\lambda_o,W_1)\Big]+R_2$$

We have $\hat{\Pi}^o(2\pi i,\lambda_o,W_1)=\hat{\Pi}^o(2\pi i,\lambda_o,W_1)-\Pi(\infty,\lambda_o)/2\pi i$ and thus (29) becomes

$$\Pi(\tau,\lambda(\cdot))=\Pi(\infty,\lambda_0)+\tfrac{1}{2}\varepsilon\lambda_0\left[e^{-2\pi i\tau}\hat{\Pi}^0(-2\pi i,\lambda_0,W_1)-\frac{\Pi(\infty,\lambda_0)}{-2\pi i}e^{-2\pi i\tau}\right.$$

$$\left.+\,e^{2\pi i\tau}\,\hat{\Pi}^0(2\pi i,\lambda_0,W_1)-\frac{\Pi(\infty,\lambda_0)}{2\pi i}e^{2\pi i\tau}\right]+R_2.$$

Let $\Pi^0(2\pi i,\lambda_0,W_1) = x+iy$. Then we can simplify the above equation to

$$(30)\quad \Pi(\tau,\lambda(\cdot)) = \Pi(\infty,\lambda_0) +\varepsilon\lambda_0\{x\cos 2\pi\tau-y\sin 2\pi\tau-\Pi(\infty,\lambda_0)\frac{\sin 2\pi\tau}{2\pi}\}+ R_2$$

The values of x and y will depend on the specific storage process we are interested in. Both the input size distribution and the release rule are essential ingredients. We will explore this more extensively in Chapters 4 and 5. There we treat the release rules $r(x)=c$, $r(x)=a+bx$ and $r(x)=cx$, $x>0$. The expected content will also be examined in these cases.

CHAPTER 4

LIMITING RESULTS FOR THE STANDARD STORE

The standard release rule in storage theory is $r(x)=c$, $x>0$. For such a process, the content of the store is released at a constant rate as long as the store is not empty. In this chapter we will investigate at some length this storage process. We focus on the particular case of nonhomogeneous Compound Poisson input with periodic intensity $\lambda(t)=\lambda_0(1+\epsilon\phi(t))$ where $\int_0^1\phi(u)du=0$. Without loss of generality we assume that $c=1$. A simple change of time scale can transform a system with $c\neq 1$ to one with $c=1$.

4.1 The Limiting Probability of Emptiness

A store with $r(x)=1$ is a Type A storage process, since $R(x,0) = \int_0^x dy = x < \infty$ for all $x < \infty$. Thus Theorem 2.3 can be used to establish the existence of a quasi-limiting distribution of store content. We need only assume that $\beta_1\lambda_0 = \beta_1 \int_0^1\lambda(u)du < 1$.

In order to determine limiting values of either the probability of an empty store or the mean store content we will need to look at the homogeneous store. The approximations derived in Chapter 3 require values for $\Pi(\infty,\lambda_0,W)$, $\mu(\infty,\lambda_0,W)$ and $\Pi^0(z,\lambda_0,W)$. The first two quantities are standard results in storage theory: $\Pi(\infty,\lambda_0,W) = 1-\lambda_0\beta_1$ and $\mu(\infty,\lambda_0,W) = \lambda_0\beta_2/2(1-\lambda_0\beta_1)$. (See Appendix.)

A value for the Laplace Transform $\Pi^0(z,\lambda_0,W)$ can be determined through the aid of the differential equation for the Laplace-Stieltjes

Transform of store content. From Lemma 3.1 this equation is

$$(1)\quad \frac{\partial M(s,t,\lambda_o,W)}{\partial t} + \lambda_o(1-B^*(s))M(s,t,\lambda_o,W) = sM(s,t,\lambda_o,W)-s\Pi(t,\lambda_o,W)$$

Taking Laplace Transforms, with respect to t, of both sides of equation (1) yields

$$zM^o(s,z,\lambda_o,W)-M(s,0,\lambda_o,W) + \lambda_o(1-B^*(s))M^o(s,z,\lambda_o,W)$$

$$= sM^o(s,z,\lambda_o,W)-s\Pi^o(z,\lambda_o,W)$$

or

$$(2)\quad M^o(s,z,\lambda_o,W) = \frac{M(s,0,\lambda_o,W)-s\Pi^o(z,\lambda_o,W)}{z-s + \lambda_o(1-B^*(s))}$$

Since $M(s,t,\lambda_o,W)$ is a Laplace-Stieltjes Transform it is analytic for $R(s)>0$. The Laplace Transform of M is also analytic in the right half plane, because an analytic function of an analytic function is analytic. Thus any zero in $R(s)>0$ of the denominator in (2) must also be a zero of the numerator.

Let $\psi(s)=z-s + \lambda_o(1-B^*(s))$. We will use Rouches Theorem to show that $\psi(s)$ has a zero in $R(s)\geq 0$. Fix z, $R(z)\geq 0$. Define an increasing sequence of real numbers γ_k, $0 < \gamma_k < 1$ such that $\gamma_k \uparrow 1$. For each integer k, let Γ_k be the contour $\{u: |u-a| = \lambda_o-\varepsilon_k\}$ where $a=\lambda_o+z$ and ε_k is chosen such that $\varepsilon_k<\lambda_o(1-\gamma_k)$. We also let $f(s)=z-s+\lambda_o$ and $g_k(s) = -\lambda_o\gamma_k B^*(s)$.

Clearly, both f and g_k are analytic in and on Γ_k. On Γ_k, $|f(s)| = \lambda_o - \varepsilon_k$ and $|g_k(s)| \le \lambda_o\gamma_k \mid B^*(s) \mid \le \lambda_o\gamma_k$. By assumption

$$|f(s)| = \lambda_o - \varepsilon_k > \lambda_o\gamma_k \ge |g_k(s)|$$

Hence by Rouches Theorem, the number of zeros of f in Γ_k is equal to the number of zeroes of $f+g_k$ in Γ_k. There is only one zero of f in Γ_k (at $s=\lambda_o+z$). Thus there is one zero of $f+g_k$ in Γ_k. Call the zero s_k.

Let Γ be the contour $\{u: | u-a | = \lambda_o\}$. Certainly s_k is inside Γ. Since $\{s_k\}$ is a bounded infinite sequence there is an s^* and a subsequence s_{k_n} such that $s_{k_n} \to s^*$ as $n\to\infty$. The point s^* is in or on Γ. We want to show that $\psi(s^*)=0$. Let $g(x) = -\lambda_o B^*(s) = \lim_{k\to\infty} g_k(x)$. We have

$$\psi(s_{k_n}) = (f + g_{k_n} - g_{k_n} + g)(s_{k_n})$$

$$= (f + g_{k_n})(s_{k_n}) + (g - g_{k_n})(s_{k_n})$$

$$= (g - g_{k_n})(s_{k_n})$$

Now

$$(g - g_{k_n})(s_{k_n}) = -(1-\gamma_{k_n})\,\lambda_o\, B^*(s_{k_n})$$

and

$$|(g - g_{k_n})(s_{k_n})| \le (1-\gamma_{k_n})\,\lambda_o \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$\lim_{n\to\infty} \psi(s_{k_n}) = 0 = \psi(s^*);$$

i.e. there is a zero of $\psi(s)$ in or on Γ.

If $R(s^*)>0$ then $M^0(s^*,z,\lambda_0,W) < \infty$, so we must have $M(s^*,0,\lambda_0,W) - s^* \Pi^0(z,\lambda_0,W) = 0$. If $R(s^*) = 0$ then we must check to see if $M^0(s^*,z,\lambda_0,W) < \infty$. But $|M(s^*,t,\lambda_0,W)| \leq 1$, so $M^0(s^*,z,\lambda_0,W) \leq \frac{1}{z} < \infty$, and again we must have $M(s^*,0,\lambda_0,W) - s^* \Pi^0(z,\lambda_0,W) = 0$. Thus we have the relationship:

$$(3) \qquad \Pi^0(z,\lambda_0,W) = \frac{M(s^*,0,\lambda_0,W)}{s^*}$$

The actual value of Π^0 will depend on the initial distribution of content. Looking at equation 3.26 we see that the distribution of interest is $W_1(x) = \tilde{W}(x) * B(x)$ where $\tilde{W}$ is the limiting distribution of content of the homogeneous store, intensity λ_0, and B is the distribution of input size. For every t, $M(s,t,\lambda_0,\tilde{W}) = M(s,\infty,\lambda_0,\tilde{W})$. Thus $M(s,0,\lambda_0,W_1) = M(s,\infty,\lambda_0,\tilde{W}) * B^*(s)$. Smith (1953) showed that

$$(4) \qquad M(s,\infty,\lambda_0,\tilde{W}) = \frac{s(1-\lambda_0\beta_1)}{s-\lambda_0(1-B^*(s))}$$

Recalling that $z - s^* + \lambda_0(1-B^*(s^*)) = 0$, we can now write

$$\Pi^0(z,\lambda_0,W_1) = \frac{(1-\lambda_0\beta_1)\, B^*(s^*)}{z}$$

The value of $B^*(s^*)$ can be computed in two ways. The most obvious way is to find s^*, the zero of $\psi(s)$. Another, more illuminating way, is to consider the following. Let $G(x,\lambda_0)$ be the distribution of the busy period. It is well known that the relationship

$$(5) \qquad G^*(z,\lambda_0) = B^*(z+\lambda_0 G^*(z,\lambda_0))$$

holds. Let $u = z+\lambda_0-\lambda_0 G^*(z,\lambda_0)$. Then $G^*(z,\lambda_0) = \dfrac{u - z - \lambda_0}{-\lambda_0}$.

Substituting into (5) gives

$$B^*(u) = \frac{u - z - \lambda_0}{-\lambda_0} .$$

or

$$(6) \qquad -\lambda_0 B^*(u) + \lambda_0 + z - u = 0$$

A zero of (6) is $u = s^*$. Thus

$$(7) \qquad B^*(s^*) = G^*(z,\lambda_0) .$$

Returning to the original problem, we can now write

$$\Pi^0(z,\lambda_0,W_1) = \frac{(1 - \lambda_0\beta_1)\ G^*(z,\lambda_0)}{z}$$

$$= (1 - \lambda_0\beta_1)G^0(z,\lambda_0)$$

and we thus have

$$\Pi(t,\lambda_0,W_1) = (1 - \lambda_0\beta_1)G(z,\lambda_0) .$$

Finally, we can write down an equation for the limiting value of the probability of emptiness as

(8) $$\lim_{n\to\infty} \Pi(n+\tau,\lambda(\cdot),\tilde{W})=(1-\lambda_o\beta_1)+\varepsilon\lambda_o(1-\lambda_o\beta_1)\sum_{m=-M}^{M} a_m e^{2\pi im\tau}\frac{[G*(2\pi im)-1]}{2\pi im}+O(\varepsilon^2)$$

where $\phi(t) = \sum_{m=-M}^{M} a_m e^{2\pi imt}$.

4.2 The Mean Store Content

An approximation to the mean store content can be obtained by using the storage equation:

(9) $$Z(t)=Z(0) + \sum_{j=o}^{N(t)} U_j - \int_o^t r(Z(u))du.$$

Taking expectations of (9) we have

$$\mu(t,\lambda(\cdot),\tilde{W}) = \mu(0,\lambda(\cdot),\tilde{W}) + \beta_1 EN(t) - \int_o^t (1-\Pi(u,\lambda(\cdot),\tilde{W}))du$$

$$= \mu(0,\lambda(\cdot),\tilde{W}) + \beta_1 \int_o^t \lambda(u)du - t + \int_o^t \Pi(u,\lambda(\cdot),\tilde{W})du$$

(10) $$= \mu(0,\lambda(\cdot),\tilde{W}) + \beta_1\lambda_o t + \beta_1\lambda_o\varepsilon\Phi(t) - t + \int_o^t \Pi(u,\lambda(\cdot),\tilde{W})du$$

We can substitute for $\Pi(u,\lambda(\cdot),\tilde{W})$ into equation (10) to obtain

$$\mu(t,\lambda(\cdot),\tilde{W}) = \mu(0,\lambda(\cdot),\tilde{W}) + (\beta_1\lambda_o-1)t + \varepsilon\lambda_o\beta_1\Phi(t) + \int_o^t \Pi(\infty,\lambda_o)du$$

$$+ \varepsilon\lambda_o \sum_{m=-M}^{M} a_m \int_o^t e^{2\pi imu}\int_o^u e^{-2\pi imx}\, \hat{\Pi}(x,\lambda_o,W_1)dxdu + O(\varepsilon^2)$$

Writing $\Pi(\infty,\lambda_o) = 1-\lambda_o\beta_1$ and $\mu(0,\lambda(\cdot),\tilde{W}) = \mu(\infty,\lambda_o)$ and noting that

$$\int_o^t\int_o^u e^{2\pi im(u-x)}\hat{\Pi}(x,\lambda_o,W_1)dxdu = \int_o^t \hat{\Pi}(x,\lambda_o,W_1)\left[\frac{e^{2\pi im(t-x)}-1}{2\pi im}\right]dx ,$$

AD-A069 550 NORTH CAROLINA UNIV AT CHAPEL HILL INST OF STATISTICS F/G 12/1
QUEUES AND STORES WITH NONHOMOGENEOUS INPUT.(U)
APR 79 D K MCCLISH N00014-76-C-0550
UNCLASSIFIED MIMEO SER-1225 NL

2 OF 2
AD A069550

END
DATE
FILMED
7-79
DDC

we have

$$(11)\quad \mu(t,\lambda(\cdot),\tilde{W})=\mu(\infty,\lambda_o)+\varepsilon\lambda_o\beta_1\Phi(t)+\varepsilon\lambda_o\sum_{m=-M}^{M}\frac{a_m}{2\pi im}\int_0^t\hat{\Pi}(x,\lambda_o,W_1)\left[e^{2\pi im(t-x)}-1\right]dx+O(\varepsilon^2)$$

Write $\Phi(t)=\sum_{m=-M}^{M}\alpha_m e^{2\pi imt}$. Note that the following identities hold:

$$\alpha_k=\begin{cases} a_m/2\pi im & k\neq 0 \\ -\sum_{m=-M}^{M}(a_m/2\pi im) & k=0. \end{cases}$$

Then

$$\mu(t,\lambda(\cdot),\tilde{W})=\mu(\infty,\lambda_o)+\varepsilon\lambda_o\beta_1\Phi(t)+\varepsilon\lambda_o\sum_{\substack{m=-M\\ m\neq 0}}^{M}\alpha_m\int_0^t e^{2\pi im(t-k)}\hat{\Pi}(x,\lambda_o,W_1)dx$$

$$+\varepsilon\lambda_o\alpha_o\int_0^t\hat{\Pi}(x,\lambda_o,W_1)du+O(\varepsilon^2).$$

As in Chapter 3, we have for $m\neq 0$,

$$\lim_{n\to\infty}\int_0^{n+\tau}e^{2\pi im(n+\tau-x)}\hat{\Pi}(x,\lambda_o,W_1)dx=e^{2\pi i\tau}\hat{\Pi}^o(2\pi im,\lambda_o,W_1)$$

$$=e^{2\pi im\tau}(1-\lambda_o\beta_1)(G^*(z,\lambda_o)-1)/z.$$

When $m=0$

$$\lim_{n\to\infty}\int_0^{n+\tau}\hat{\Pi}(x,\lambda_o,W_1)dx=\int_0^{\infty}\hat{\Pi}(x,\lambda_o,W_1)dx$$

$$=\int_0^{\infty}(1-\lambda_o\beta_1)(G(x,\lambda_o)-1)dx$$

$$=-(1-\lambda_o\beta_1)\gamma_1$$

where γ_1 is the expected length of the busy period. Thus we can finally write

$$\lim_{n\to\infty}\mu(n+\tau,\lambda(\cdot),\tilde{W}) = \mu(\infty,\lambda_o)+\varepsilon\lambda_o\beta_1\Phi(\tau)-\varepsilon\lambda_o(1-\lambda_o\beta_1)\gamma_1\alpha_o$$

$$(12)\qquad + \varepsilon\lambda_o(1-\lambda_o\beta_1)\sum_{\substack{m=-M\\ m\neq o}}^{M}\alpha_m \frac{e^{2\pi i m\tau}(G^*(2\pi i m,\lambda_o)-1)}{2\pi i m} + O(\varepsilon^2)$$

We shall henceforth denote the approximations to the limits in (8) and (12) by $\tilde{\Pi}(\tau,\lambda(\cdot))$ and $\tilde{\mu}(\tau,\lambda(\cdot))$.

It does not follow immediately that the limit of $\mu(n+\tau,\lambda(.),W)$ is equal to the expectation of the quasi-limiting random variable. It is still necessary to prove a condition such as uniform integrability. We do this by proving uniform integrability for a sequence of random variables which bounds our store content from above. According to Lemma 3.2.1, for two processes with the same release rule and Compound Poisson input

$$\mu_k(t,\lambda(\cdot),W) \leq \mu_k(t,\lambda_o(1+\varepsilon M),W)$$

where $\phi(t) \leq M$ and $\mu_k(t,\lambda(\cdot),W)$ denotes the k^{th} moment of store content at time t of a store with intensity $\lambda(\cdot)$ and initial distribution W. If we can show that $\sup_n \mu_2(n+\tau,\lambda_o(1+\varepsilon M),W)<\infty$ then $\sup_n \mu_2(n+\tau,\lambda(\cdot),W)<\infty$, which implies uniform integrability.

In standard queueing and storage theory it has been shown that as long as $\lambda_o(1+\varepsilon M)\beta_1<1$ and $\beta_2<\infty$ then, if $\lambda_1=\lambda_o(1+\varepsilon M)$,

$$(13)\qquad \lim_{n\to\infty} \mu_1(n+\tau,\lambda_1 \text{[illegible]} = \lambda_1\beta_2/2(1-\lambda_1\beta_1)$$

where $\lambda_1\beta_2/2(1-\lambda_1\beta_1)$ is the expectation of the limiting value of store content. Since store content is nonnegative, (13) implies uniform integrability for the homogeneous process. Thus

$$\sup_n \mu_2(n+\tau,\lambda(\cdot),W) \le \sup_n \mu_2(n+\tau,\lambda_1,W) < \infty$$

and we have uniform integrability of the nonhomogeneous process. This implies the anticipated conclusion.

4.3 Special Case: $\phi(t) = \cos 2\pi t$

Let us now look at a specific example in more detail. Suppose $\phi(x)=\cos 2\pi x = \frac{1}{2} e^{-2\pi i x} + \frac{1}{2} e^{2\pi i x}$. Equation (8) becomes

$$(14)\quad \hat{\Pi}(\tau,\lambda(\cdot))=(1-\lambda_o\beta_1)+\varepsilon\lambda_o(1-\lambda_o\beta_1)\left\{\frac{1}{2} \frac{e^{-2\pi i\tau}G^*(-2\pi i,\lambda_o)}{-2\pi i} - \frac{1}{2}\frac{e^{-2\pi i\tau}}{-2\pi i} + \frac{1}{2} \frac{e^{2\pi i\tau}G^*(2\pi i,\lambda_o)}{2\pi i} - \frac{1}{2}\frac{e^{2\pi i\tau}}{2\pi i}\right\}$$

Equation (14) has an alternative form without any complex numbers. Let $G^*(2\pi i,\lambda_o)=x+iy$. Then $G^*(-2\pi i,\lambda_o)=x-iy$ and (14) can be written

$$(15)\quad \tilde{\Pi}(\tau,\lambda(.))=(1-\lambda_o\beta_1)+\frac{\varepsilon\lambda_o(1-\lambda_o\beta_1)}{2\pi}\{x\sin 2\pi\tau+y\cos 2\pi\tau-\sin 2\pi\tau\}$$

To determine $\tilde{\mu}(\tau,\lambda(\cdot))$ in this case, first note that $\phi(\tau) = \frac{1}{4\pi i}\left[e^{2\pi i\tau} - e^{-2\pi i\tau}\right]$. Since $\alpha_o=0$ we do not need to know the expected length of the busy period here. The approximation to the mean in this particular case is:

$$\tilde{\mu}(\tau,\lambda(\cdot))=\frac{\lambda_o\beta_2}{2(1-\lambda_o\beta_1)} +\varepsilon\lambda_o\beta_1\frac{\sin 2\pi\tau}{2\pi}+\varepsilon\lambda_o(1-\lambda_o\beta_1)\left\{\frac{1}{4\pi i}\frac{e^{-2\pi i\tau}G^*(-2\pi i)}{-2\pi i} - \frac{\left(-\frac{1}{4\pi i}\right)e^{-2\pi i\tau}}{-2\pi i} + \frac{1}{4\pi i}\frac{e^{2\pi i\tau}G^*(2\pi i)}{2\pi i} - \frac{\frac{1}{4\pi i}e^{2\pi i\tau}}{2\pi i}\right\}$$

$$(16) \quad = \frac{\lambda_o\beta_2}{2(1-\lambda_o\beta_1)} - \frac{\varepsilon\lambda_o(1-\lambda_o\beta_1)(x-1)\cos 2\pi\tau}{4\pi^2} + \frac{\varepsilon\lambda_o}{4\pi^2}\left[2\pi\beta_1+(1-\lambda_o\beta_1)y\right]\sin 2\pi\tau$$

4.4 A Queueing Theory Example

The actual value of $\hat{\Pi}$ and $\tilde{\mu}$ depend on the distribution $B(x)$. Suppose that we have the most basic queueing situation-negative exponential service time with average service time $1/\mu$. It is well known that the Laplace-Stieltjes Transform of the busy period of an M/M/1 queue is

$$G^*(z,\lambda_o) = \frac{\mu + \lambda_o + z - \sqrt{(\mu+\lambda_o+z)^2-4\lambda_o\mu}}{2\lambda_o}$$

and the mean waiting time in steady state is $\lambda_o/\mu(\mu-\lambda_o)$.

Figures 4.1-4.3 present graphs of the approximation (15) of the limiting probability of emptiness in an M(t)/M/1 queue with $\lambda(t) = \lambda_o(1+\varepsilon\cos 2\pi t)$. We take $\mu=10$ for all three figures, while $\lambda_o = 1.0, 5.0, 9.5$ in Figures 4.1-4.3 respectively. In each figure, ε takes on the values .1, .25, .5 and .9.

The values of $\tilde{\Pi}(\infty,\lambda(\cdot))$ fluctuate symmetrically about $\Pi(\infty,\lambda_o) = 1-\lambda_o/\mu$, for each ε, with the amount of fluctuation dependent on the size of ε. This was to be expected from the form of equation (15). It is also intuitively reasonable that the probability should vary about the emptiness probability of the M/M/1 queue which uses the average intensity per period as the arrival intensity. In fact, the M(t)/M/1 queue with $\lambda(t) = \lambda_o(1+\varepsilon\phi(t))$ is usually approximated by the M/M/1 queue with $\lambda(t)\equiv\lambda_o$.

It is interesting to note how the extreme values of $\tilde{\Pi}(\tau,\lambda(\cdot))$ differ from $\Pi(\infty,\lambda_o)$. This gives an indication of how much the process is smoothed by using a homogeneous intensity rather than a nonhomogeneous one. The extrema of $\tilde{\Pi}(\tau,\lambda(\cdot))$ can be determined by differentiating (15) with respect to τ:

$$\frac{\partial\tilde{\Pi}(\tau,\lambda(\cdot))}{\partial\tau} = \varepsilon\lambda_o(1-\lambda_o\beta_1)2\pi\{x\cos 2\pi\tau - y\sin 2\pi\tau - \cos 2\pi\tau\}.$$

Setting the derivative equal to zero, we find that the extrema occur at points τ_o in the period such that

$$\tan 2\pi\tau_o = (x-1)/y.$$

Note that the extrema occur at the same point in each period, irrespective of the value of ε, which only effects the value at that point. The extrema for each graph are shown in Table 4.1. If we consider the percentage deviation of the extrema from $\Pi(\infty,\lambda_o)$, we see that the percentages increase as λ_o increases. These values are in Table 4.1.

It is also of interest to note the time from the maximum (minimum) value of $\lambda(t)$ to the corresponding minimum (maximum) value of $\tilde{\Pi}(\tau,\lambda(\cdot))$. We shall call this the response time, as it measures how long it takes for the queue to fully respond to extremes in the arrival pattern. Values of response time are found in Table 4.1. Note that as λ_o increases, the response time increases.

Table 4.1

λ_o	Extreme Values	% Deviation from $\Pi(\infty,\lambda_o)$	Response Time
1.0	$.90 \mp .08\varepsilon$	9ε	.0995
5.0	$.50 \mp .23\varepsilon$	46ε	.1355
9.5	$.05 \mp .045\varepsilon$	90ε	.1655

FIGURE 4.1

Probability of an Empty Queue

$r(x)=1$

$\lambda_0=1.0 \quad \mu=10.0$

$\lambda(t)=\lambda_0(1+\varepsilon\cos 2\pi t)$

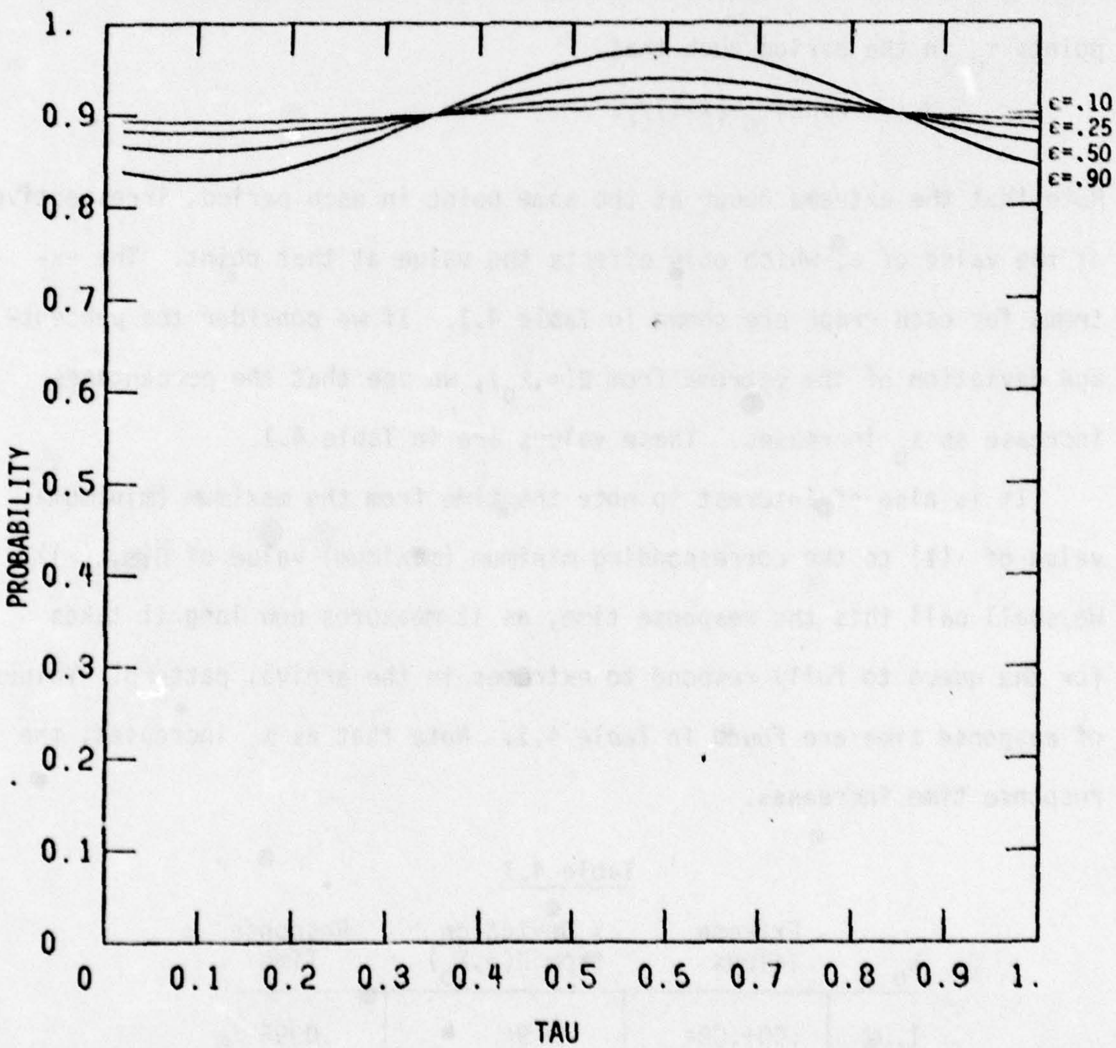


FIGURE 4.2

Probability of an Empty Queue

$r(x)=1$

$\lambda_o=5.0 \quad \mu=10.0$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$

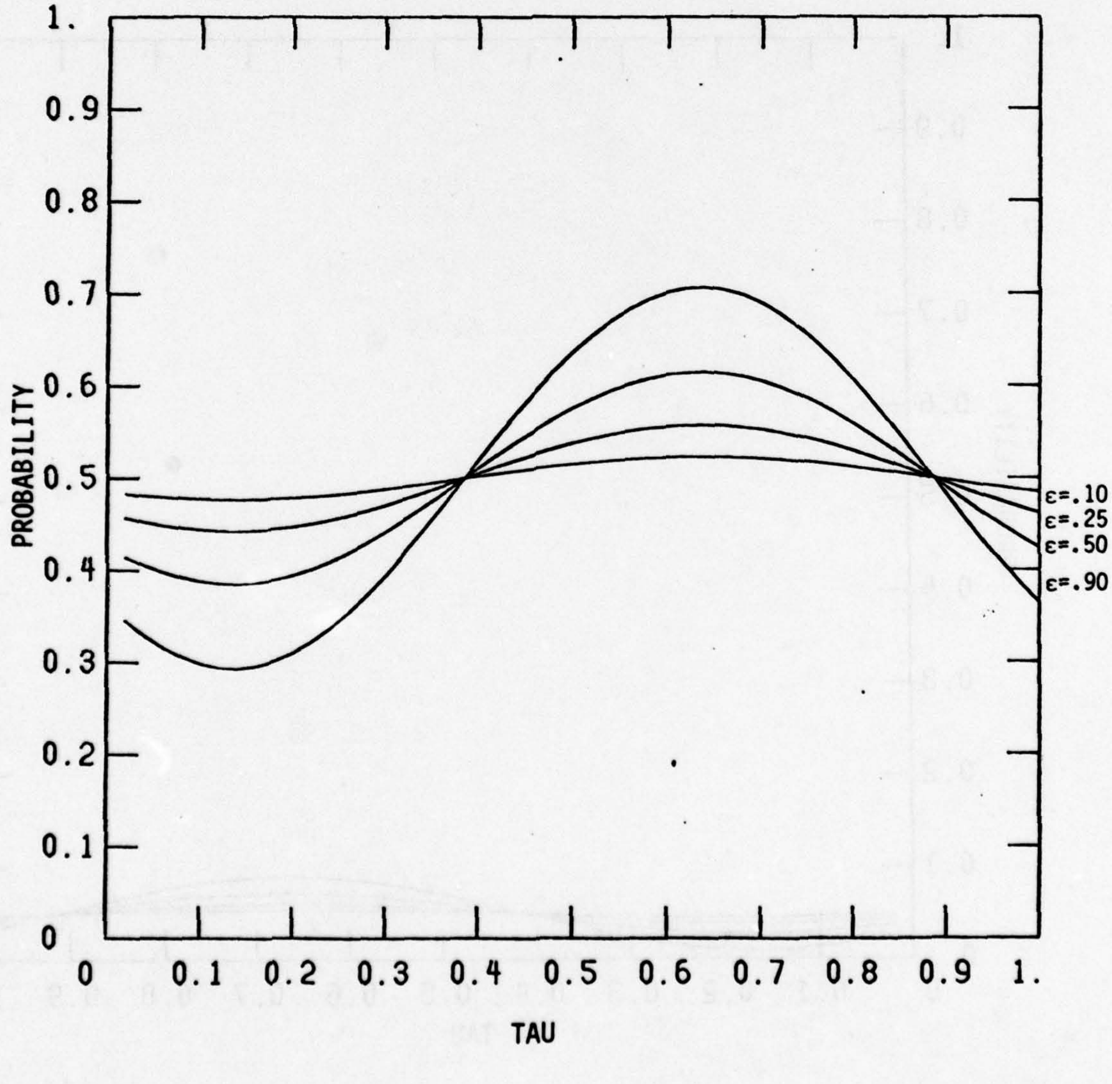


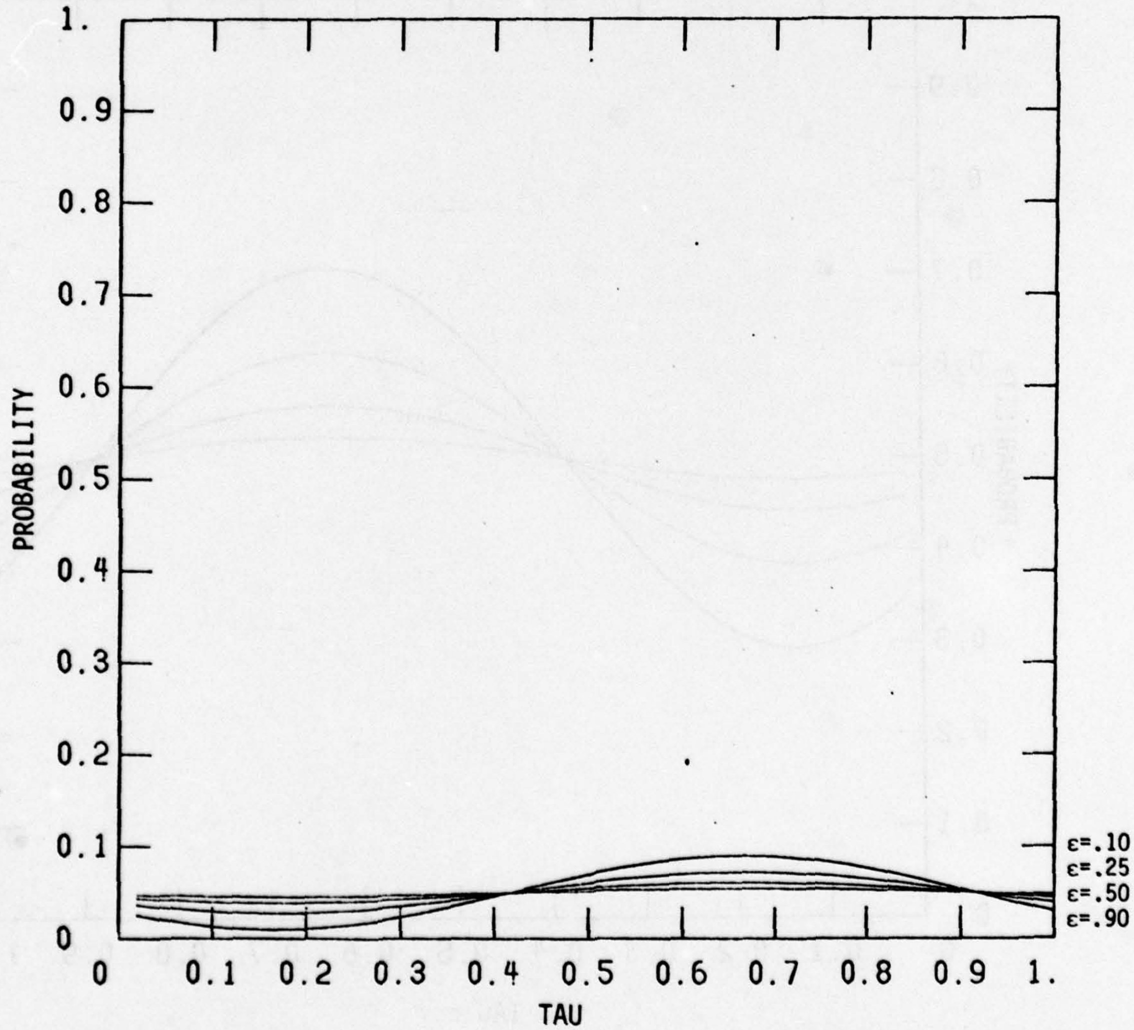
FIGURE 4.3

Probability of an Empty Queue

$r(x)=1$

$\lambda_o=9.5 \quad \mu=10.0$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$



Figures 4.4-4.6 are graphs of the mean waiting time of the M(t)/M/1 queue, $\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$. Again, we let $\mu=10$ and $\lambda_o=1.0$, 5.0, 9.5 respectively for the three figures, and ε takes on the values .1, .25, .5, .9. The graphs of the mean waiting time are similar to those of the emptiness probability. The values of $\tilde{\mu}(\tau,\lambda(\cdot))$ fluctuate symmetrically about $\mu(\infty,\lambda_o)$. The extrema occur at points τ_o such that

$$\tan 2\pi\tau_o=\frac{2\pi\beta_1+(1-\lambda_o\beta_1)y}{(1-\lambda_o\beta_1)(1-x)}\quad .$$

Table 4.2 gives the extrema, percentage deviation of the extrema from $\mu(\infty,\lambda_o)$, and the response time for each graph.

Table 4.2

λ_o	Extreme Values	% Deviation from $\mu(\infty,\lambda_o)$	Response Time
1.0	$.011\pm.009\varepsilon$	82ε	.1071
5.0	$.10\ \pm.06\varepsilon$	60ε	.1768
9.5	$1.90\ \pm.14\varepsilon$	7ε	.2434

We see that while response time increases as λ_o increases, the percentage deviation of the extrema from $\mu(\infty,\lambda_o)$ decreases, in contrast to what happened with $\tilde{\Pi}(\tau,\lambda(\cdot))$.

4.5 General Service Distribution—An Approximation

As mentioned earlier in the chapter, to determine explicitly the value of $\tilde{\Pi}(\tau,\lambda(\cdot))$ and $\tilde{\mu}(\tau,\lambda(\cdot))$ for a store, we need to either know the busy period distribution or be able to find the zero of the equation $z+\lambda(1-B^*(u))-u=0$, $R(u)\geq 0$. This may not be an easy task for an arbitrary distribution. An approximation to many general distributions which often

FIGURE 4.4

Mean Waiting Time

$r(x)=1$

$\lambda_o=1.0 \quad \mu=10.0$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$

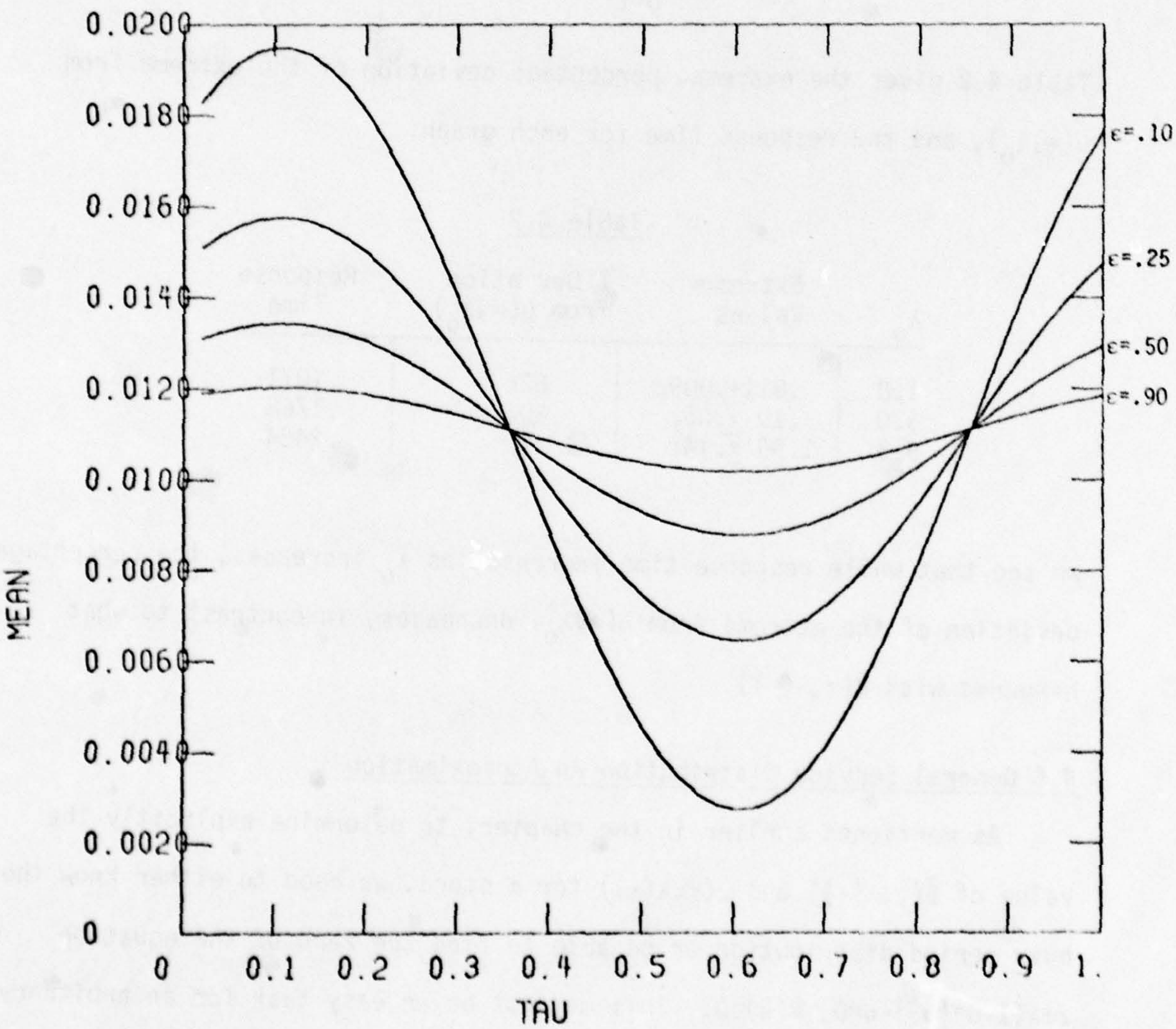


FIGURE 4.5

Mean Waiting Time

$r(x)=1$

$\lambda_0=5.0 \quad u=10.0$

$\lambda(t)=\lambda_0(1+\varepsilon\cos 2\pi t)$

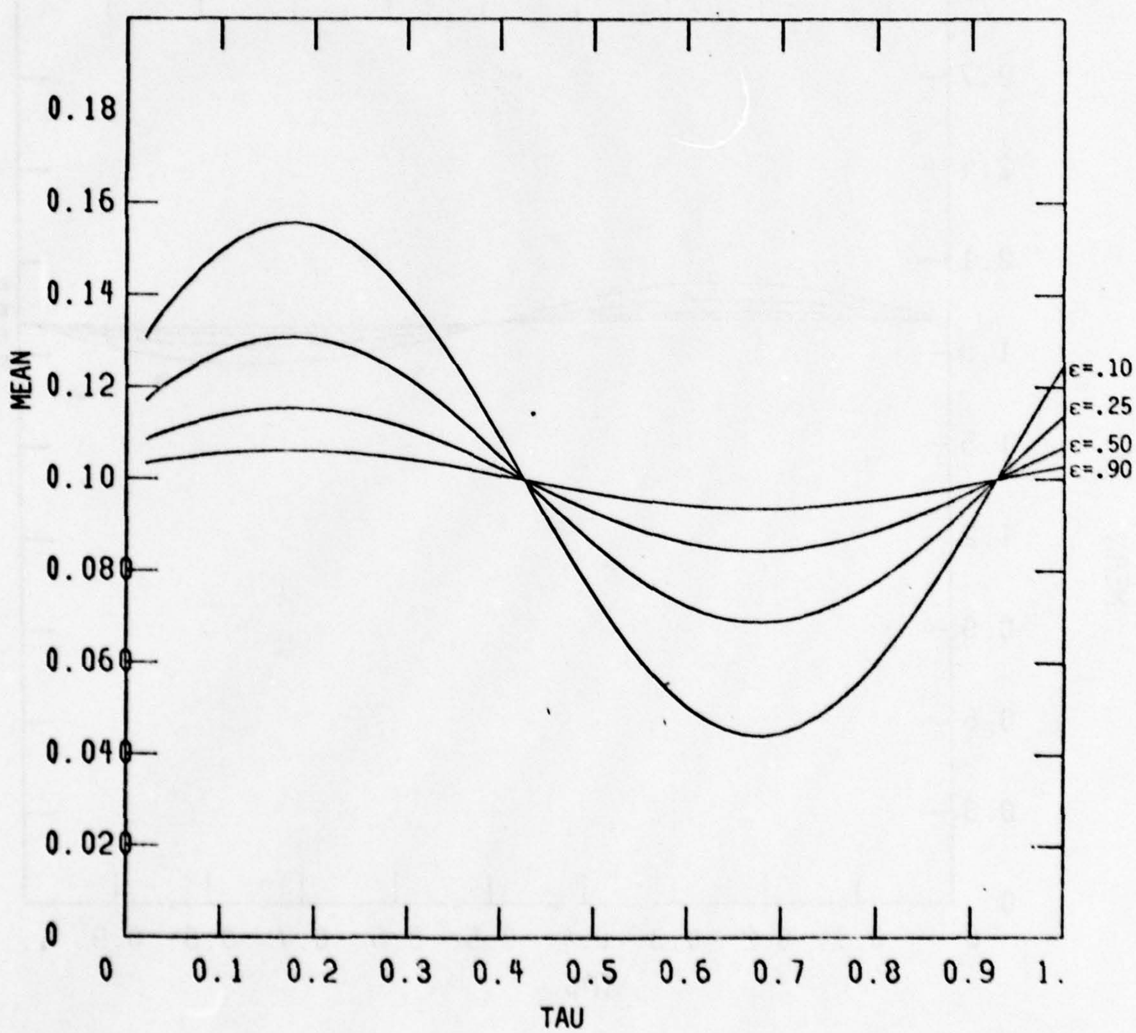


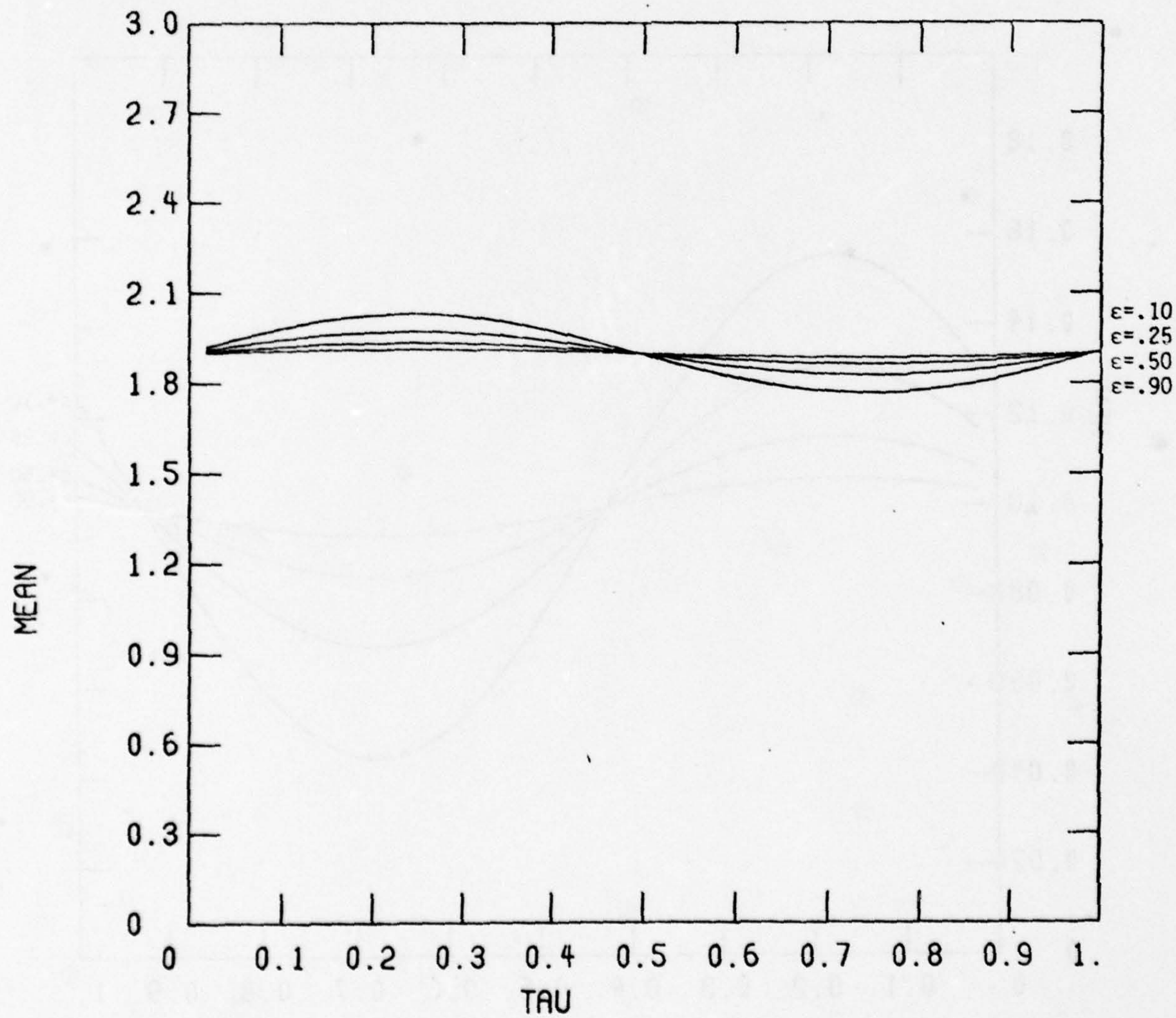
FIGURE 4.6

Mean Waiting Time

$r(x)=1$

$\lambda_0=9.5 \quad \mu=10.0$

$\lambda(t)=\lambda_0(1+\cos 2\pi t)$



is simpler to deal with is the Weighted-sum Erlang distribution which has density

$$(17)\quad B'(t)=b(t)=\sum_{k=1}^{\infty} c_k e^{-\mu t}\mu\frac{(\mu t)^{k-1}}{(k-1)!};\quad 0\le c_k\le 1,\ \sum_{k=1}^{\infty} c_k=1$$

With proper choice of $\{c_m\}$ and μ it can approximate a wide range of distributions.

The Weighted-sum Erlang distribution can be seen to be related to the more basic negative exponential distribution (which it contains as a special case, when $c_1=1$), deriving from the relationship its advantage as an approximation. This is easier to understand in the context of queueing theory. Suppose we let $\lambda(t)c_m dt+o(dt)$ be the probability that during $(t,t+dt)$ a customer requiring m phases of service arrives at the queue. If the distribution of each phase of service is negative exponential with mean $1/\mu$ then $B(t)$ as in (17) is the resulting (total) service time distribution for each customer.

It is also interesting to note that this same approach can be used to model a bulk queue. In this case $\lambda(t)c_m dt+o(dt)$ is to be interpreted as the probability that during $(t,t+dt)$ m customers will arrive at the queue, each with negative exponential service time.

The Laplace-Stieltjes Transform of $B(t)$ is easy to find. We have

$$B^*(s)=\sum_{k=1}^{\infty} c_k \int_0^{\infty} e^{-st}e^{-\mu t}\frac{(\mu t)^{k-1}}{(k-1)!}\mu dt$$

$$=\sum_{k=1}^{\infty} \frac{c_k\mu^k}{(s+\mu)^k}\int_0^{\infty} e^{-(s+\mu)t}(s+\mu)^k\frac{t^{k-1}}{(k-1)!}dt$$

$$(18)\quad =\sum_{k=1}^{\infty} c_k\frac{\mu^k}{(s+\mu)^k},\quad = C\left(\frac{\mu}{\mu+s}\right)$$

where $C(x)$ is the probability generating function of $\{c_m\}$. Since we know that $G^*(z,\lambda_o) = B^*(z+\lambda_o-\lambda_o G^*(z,\lambda_o))$ we can solve for $G^*(z,\lambda_o)$ once we determine the probability generating function of $\{c_m\}$.

4.6 A Special Case: $c_m=(1-\alpha)\alpha^{m-1}$

Suppose $c_m = (1-\alpha)\alpha^{m-1}$, $m>0$. Then

$$C(x) = \frac{x(1-\alpha)}{1-\alpha x}$$

Thus

$$G^*(z,\lambda_o) = C\left(\frac{\mu}{z+\lambda_o-\lambda_o G^*(z,\lambda_o)+\mu}\right)$$

$$= \frac{(1-\alpha)\dfrac{\mu}{z+\lambda_o-\lambda_o G^*(z,\lambda_o)+\mu}}{1-\dfrac{\alpha\mu}{z+\lambda_o-\lambda_o G^*(z,\lambda_o)+\mu}}$$

$$= \frac{\mu(1-\alpha)}{z+\lambda_o-\lambda_o G^*(z,\lambda_o)+\mu-\alpha\mu}$$

This can be transformed to the equation

$$\lambda_o\left[G^*(z,\lambda_o)\right]^2-\left[z+\lambda_o+\mu-\alpha\mu\right]G^*(z,\lambda_o)+\mu(1-\alpha)=0$$

Solving this for G^* gives

$$G^*(z,\lambda_o)=\frac{z+\lambda_o+\mu-\alpha\mu \pm \sqrt{(z+\lambda_o+\mu-\alpha\mu)^2-4\lambda_o\mu(1-\alpha)}}{2\lambda_o}$$

The correct solution is the one which makes $G^*(0,\lambda_o)=1$.

We still need to determine $\Pi(\infty,\lambda_o)$ and $\mu(\infty,\lambda_o)$ to be able to find $\tilde{\Pi}(\tau,\lambda(\cdot))$ and $\tilde{\mu}(\tau,\lambda(\cdot))$. We know that $\Pi(\infty,\lambda_o) = 1-\lambda_o\beta_1$ and

$$\beta_1 = -\frac{d}{ds} B^*(s) \mid s=0$$

$$= -\frac{d}{ds} C\left(\frac{\mu}{s+\mu}\right) \Big|_{s=0}$$

In the particular case $c_m=(1-\alpha)\alpha^{m-1}$, $m>0$,

$$\beta_1 = -\frac{d}{ds}\left(\frac{\frac{\mu}{s+\mu}(1-\alpha)}{1-\frac{\alpha\mu}{s+\mu}}\right)_{s=0}$$

$$= -\left(-\frac{\mu(1-\alpha)}{(s+\mu-\alpha\mu)^2}\right)_{s=0}$$

$$= \frac{1}{\mu(1-\alpha)}$$

Thus

$$\Pi(\infty,\lambda_o) = 1 - \frac{\lambda_o}{\mu(1-\alpha)}$$

In order to determine $\mu(\infty,\lambda_o)$ we need to know the value of β_2, where

$$\beta_2 = \frac{d^2}{ds^2} B^*(s)\Big|_{s=0}$$

$$= \frac{d^2}{ds^2} C\left(\frac{\mu}{s+\mu}\right)\Big|_{s=0} \quad .$$

Again, for our example we have

$$\beta_2 = \frac{d}{ds}\left(- \frac{\mu(1-\alpha)}{(s+\mu-\alpha\mu)^2} \right) \Bigg|_{s=0}$$

$$= \frac{2\mu(1-\alpha)}{(s+\mu-\alpha\mu)^3} \Bigg|_{s=0}$$

$$= \frac{2}{\mu^2(1-\alpha)^2}$$

Thus

$$\mu(\infty,\lambda_0) = \frac{\lambda_0 \dfrac{2}{\mu^2(1-\alpha)^2}}{2(1-\lambda_0/\mu(1-\alpha))}$$

$$= \frac{\lambda_0}{\mu(1-\alpha)\left[\mu(1-\alpha)-\lambda_0\right]}$$

4.7 Another Example

In Figures 4.7-4.10 we have graphs of the probability of emptiness and mean waiting time of the $M^X(t)/M/1$ queue where $\lambda(t)=\lambda_0(1+\varepsilon\cos 2\pi t)$ and $c_m = \frac{2}{3}\left(\frac{1}{3}\right)^{m-1}$. We take $\mu=10$ for all figures, and $\lambda_0 = 1, 5$. We do not have any graphs for $\lambda_0 = 9.5$, since for this value of λ_0, $\lambda_0\beta_1 = 9.5 \times \frac{3}{20} > 1$, and no quasi-limiting value of waiting time exists. As before, in each figure ε takes on the values .1, .25, .5, .9.

These graphs are similar to the others we have seen. We see that the response time increases as λ_0 increases, and the percent deviation from the average increases for the probability of emptiness, but decreases for the mean waiting time. Tables 4.3 and 4.4 give these values.

Table 4.3

λ_o	Extreme Values	% Deviation from $\Pi(\infty,\lambda_o)$	Response Time
1	$.85+.093\varepsilon$	11ε	.0689
5	$.25\mp.135\varepsilon$	54ε	.1589

Table 4.4

λ_o	Extreme Values	% Deviation from $\mu(\infty,\lambda_o)$	Response Time
1	$.026+.016\varepsilon$	62ε	.1321
5	$.45\mp.10\varepsilon$	22ε	.1666

FIGURE 4.7

Probability of an Empty Queue

$r(x)=1$

$c_m=\frac{2}{3}\left(\frac{1}{3}\right)^{m-1}, m>0$

$\lambda_o=1.0 \quad \mu=10.0$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$

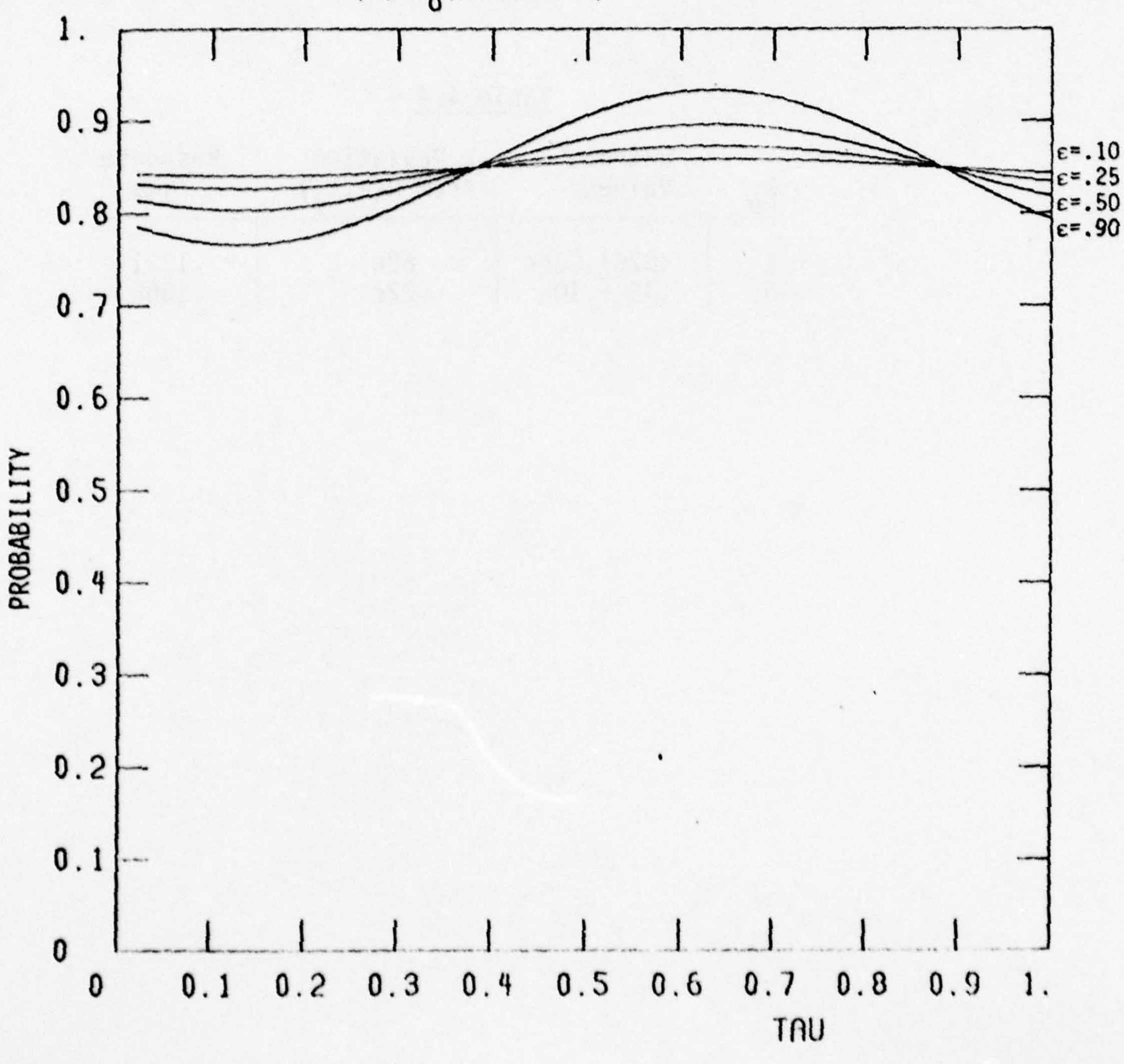


FIGURE 4.8

Probability of an Empty Queue

$r(x)=1$

$c_m=\frac{2}{3}\left(\frac{1}{3}\right)^{m-1}, m>0$

$\lambda_o=5.0 \quad \mu=10.0$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$

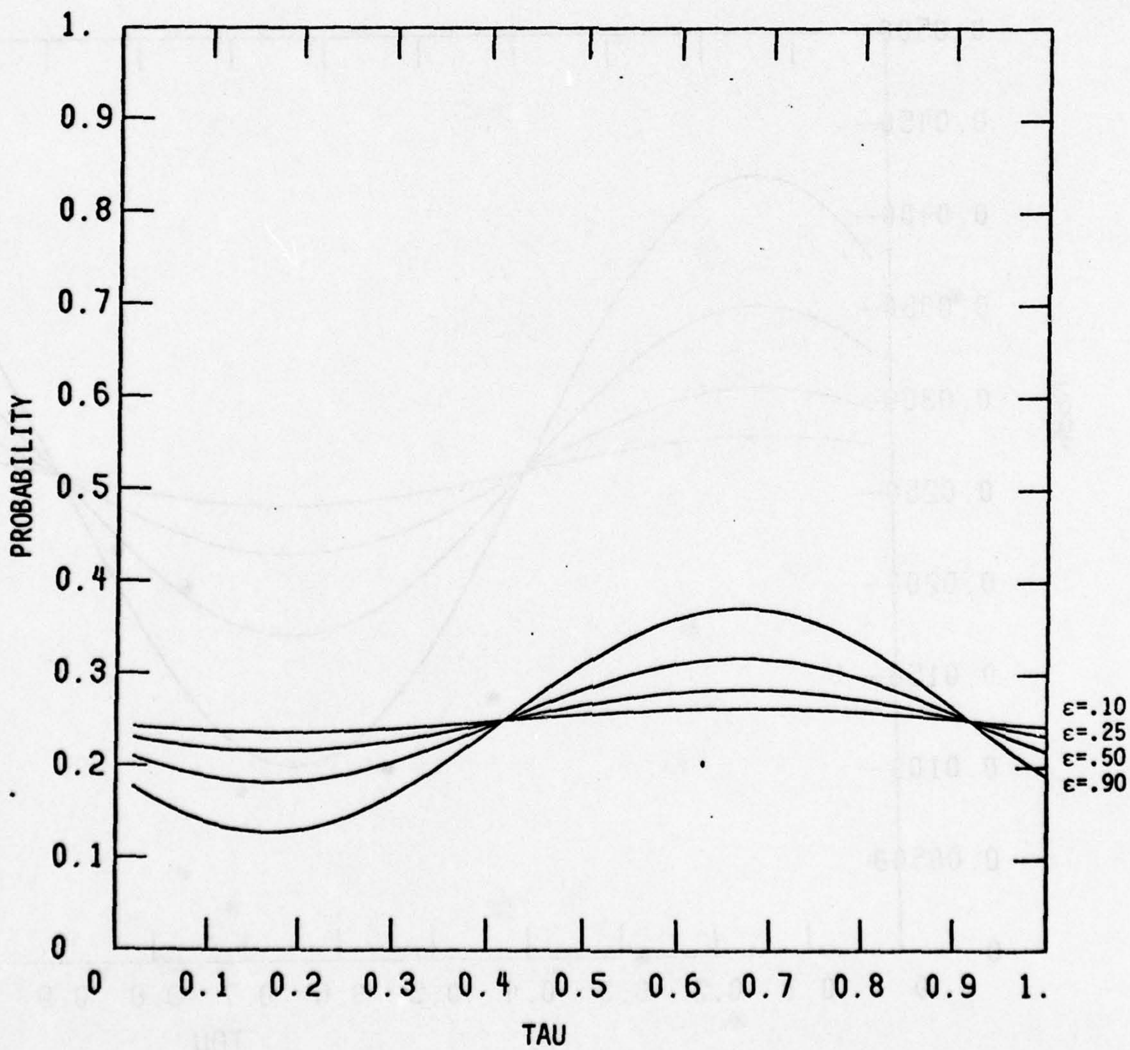


FIGURE 4.9

Mean Waiting Time

$r(x)=1$

$c_m=\frac{2}{3}\left(\frac{1}{3}\right)^m, m>0$

$\lambda_o=1.0 \quad \mu=10.0$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$

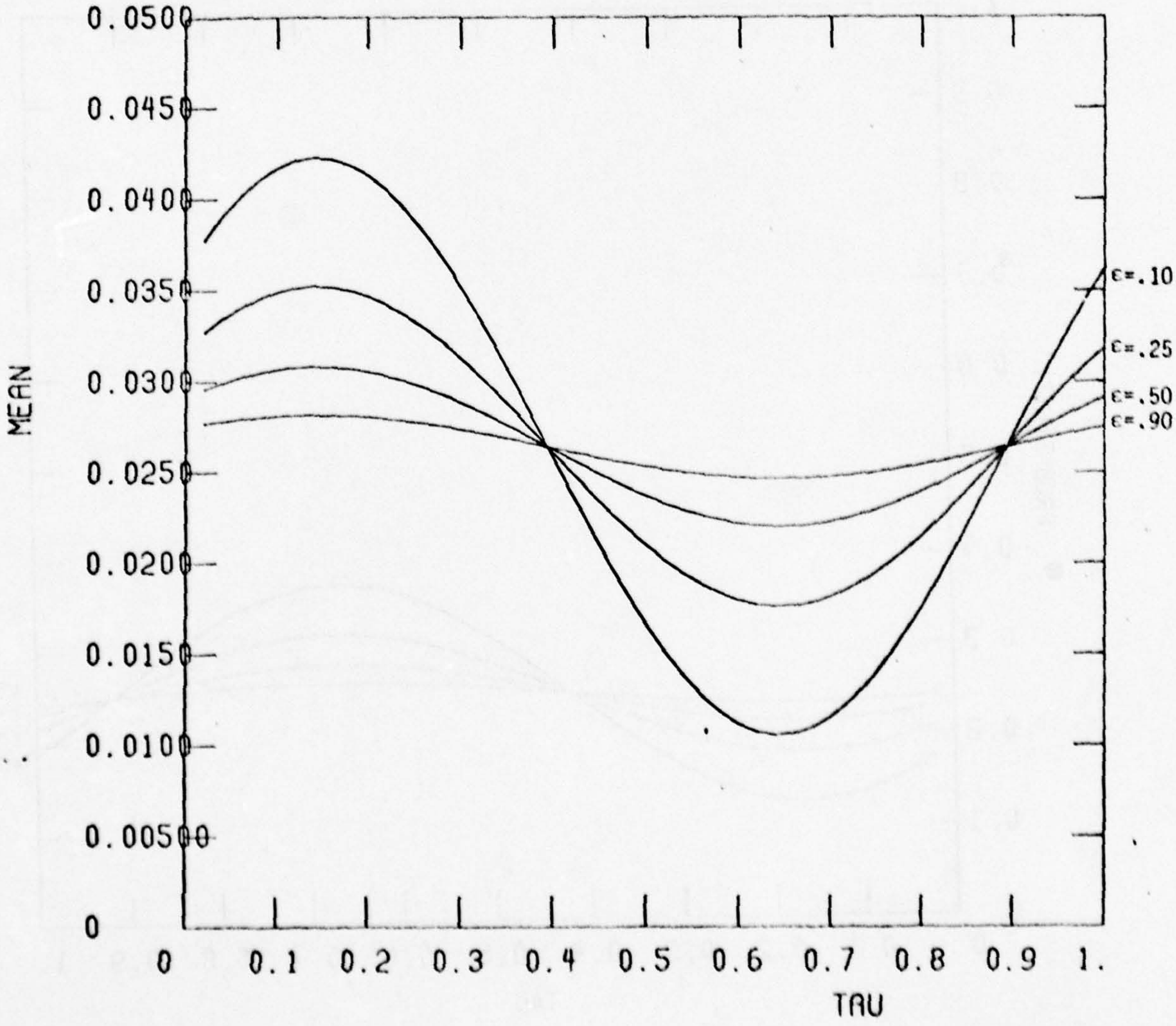


FIGURE 4.10

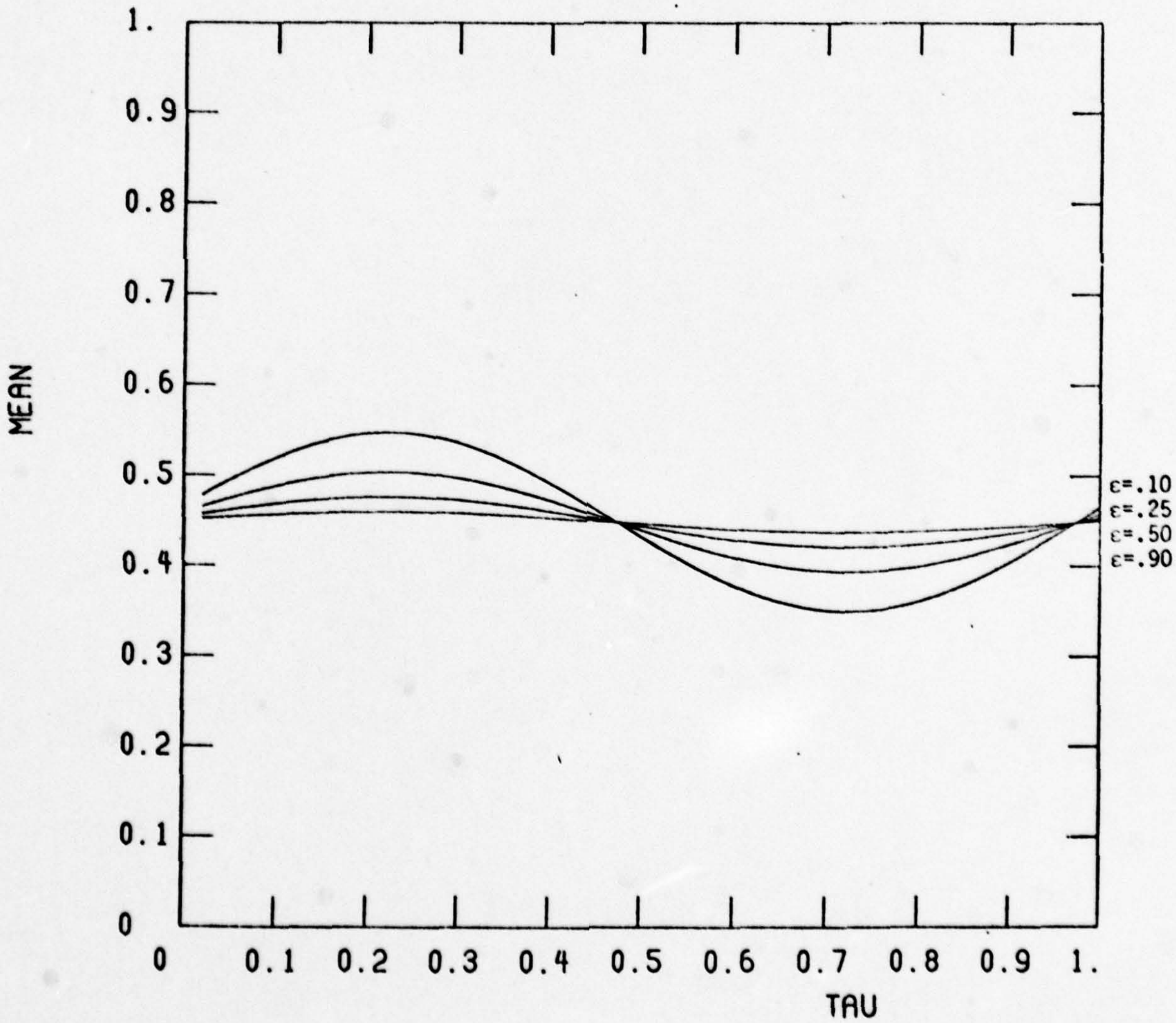
Mean Waiting Time

$r(x)=1$

$c_m=\frac{2}{3}\left(\frac{1}{3}\right)^{m-1}, m>0$

$\lambda_o=5.0 \quad \mu=10.0$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$



CHAPTER 5

LIMITING RESULTS FOR SOME STORES WITH GENERAL RELEASE RULES

There are many release rules that are of interest in storage theory. The most common, $r(x)=c, x>0$ was considered in the previous chapter. The output of such a store is not a function of the content (except that $r(Z(t))=0$ if $Z(t)=0$). In the class of "general" release rules, where output _is_ dependent on the content of the store, the release rules $r(x)=a+bx, x>0$ and $r(x)=cx, x>0$ are the most popular in the literature. Both these release rules will be discussed in this chapter.

5.1 The Limiting Probability of Emptiness; $r(x)=a+bx$

We look first at a storage process with nonhomogeneous Compound Poisson input, periodic intensity $\lambda(t)$, and linear release rate $r(x)=a+bx$; $a>0$, $b>0$. We would like to use Theorem 2.2 to show that a quasi-limiting distribution of store content exists. To do this, we need to look at $Y(u)$. Recall that

$$Y(u) = \int_0^u \frac{dx}{r(x)} \quad ,$$

$$= \int_0^u \frac{dx}{a+bx} \quad \text{, in the present case,}$$

$$= \frac{1}{b} \ln\left(1+\frac{b}{a} u\right)$$

The assumptions of Theorem 2.2 are met as long as

(1) $\left[\int_0^1 \lambda(u)du\right] E\left[\ln(1+\frac{b}{a}U)\right] < b.$

Thus we can conclude that a quasi-limiting distribution of store content exists as long as (1) holds.

Suppose $\lambda(t)=\lambda_o(1+\varepsilon\phi(t))$ where ϕ is periodic and $\int_0^1 \phi(u)du=0$. The formula for the limiting value of the probability of emptiness is dependent on the existence of a limiting distribution of the content of the corresponding homogeneous store with intensity λ_o. From Theorem 2.5 this limit will exist if there exists a level c such that $r(x)>\lambda_o\beta_1$ for $x\geq c$ and $r(x)\leq M_c$ for $x<c$. If $a>\lambda_o\beta_1$ then $r(x)\geq\lambda_o\beta_1$ for all $x>0$ and $r(x)$ is bounded on all finite intervals. If $a<\lambda_o\beta_1$ we can choose $c=\frac{1}{b}(\lambda_o\beta_1-a)$. Then $r(x)=a+bx>\lambda_o\beta_1$ for $x\geq c$ and $r(x)\leq\lambda_o\beta_1=M_c$.

To be able to compute $\Pi(\tau,\lambda(\cdot),\tilde{W})$ we need to determine $\Pi(\infty,\lambda_o,\tilde{W})$ and $\Pi^o(2\pi im,\lambda_o,W_1)$ for $m=\pm1,\pm2,\ldots,\pm M$ and $W_1(x)=\tilde{W}(x)*B(x)$ where $\tilde{W}$ is the limiting distribution of store content for the homogeneous store. An examination of the Laplace-Stieltjes Transform, $M(s,t,\lambda_o,\tilde{W})$, will be helpful. From Lemma 3.1 we know that this satisfies the following equation:

(2) $$\frac{\partial M(s,t)}{\partial t}+\left[\lambda_o(1-B^*(s))-sa\right] M(s,t)+bs\frac{\partial M(s,t)}{\partial s}=-as\Pi(t)$$

where we write $M(s,t,\lambda_o,\tilde{W})=M(s,t)$ and $\Pi(t,\lambda_o,\tilde{W})=\Pi(t)$.

Let $\tilde{M}(s)$ be the Laplace-Stieltjes Transform of the limiting value of store content. $\tilde{M}(s)$ satisfies (2) with $\partial\tilde{M}(s)/\partial t=0$. Then in the limit $(t\to\infty)$ we have the differential equation

$$\frac{d\tilde{M}(s)}{ds} + \lambda_o\left[\frac{1-B^*(s)-sa}{bs}\right]\tilde{M}(s) = -\frac{a}{b}\,\Pi(\infty).$$

The solution to this is

$$(3)\quad \tilde{M}(s) = e^{\theta(s)}\left[c-\frac{a}{b}\,\Pi(\infty)\int_0^s e^{-\theta(x)}dx\right]$$

where $\theta(s)=\frac{a}{b}\,s-\frac{\lambda_o}{b}\int_o^s \frac{1-B^*(x)}{x}\,dx$, and c is some constant. When s=0, $\tilde{M}(0)=1$. This implies that c=1. As s approaches infinity, $\tilde{M}(s)$ must approach $\Pi(\infty)$, which is finite. But $\exp(\theta(s))\to\infty$ as $s\to\infty$, so we must have that

$$\lim_{s\to\infty}\left[1-\frac{a}{b}\,\Pi(\infty)\int_0^s e^{-\theta(x)}dx\right]=0$$

or

$$(4)\quad \Pi(\infty) = \frac{b}{a\int_0^\infty e^{-\theta(x)}dx}.$$

Substituting (4) into (3) gives

$$(5)\quad \tilde{M}(s) = \frac{e^{\theta(s)}\int_s^\infty e^{-\theta(x)}\,dx}{\int_0^\infty e^{-\Theta(x)}\,dx}$$

We also note that

$$\lim_{s\to\infty} -\frac{d\tilde{M}(s)}{ds} =\mu(\infty)=\frac{a}{b}\,\Pi(\infty)-\frac{a}{b} + \lim_{s\to\infty}\frac{\lambda_o(1-B^*(s))}{bs}\,\tilde{M}(s)$$

An application of l'Hôpital's rule yields

$$(6)\quad \mu(\infty) = \frac{a\Pi(\infty) - a + \lambda_o\beta_1}{b}$$

It still remains to determine $\Pi^0(z,\lambda_0 W_1)$. We go back to equation (2), where we take the Laplace Transform with respect to t on both sides of the equation, to transform the partial differential equation into a differential equation in $M^0(s,z)$. This gives:

$$zM^0(s,z)-M(s,0)+\left[\lambda_0(1-B^*(s))-sa\right] M^0(s,z)+bs\frac{\partial M^0(s,z)}{\partial s} = -sa\Pi^0(z)$$

Reorganizing the equation into a more familiar form we have:

$$(7)\quad \frac{\partial M^0(s,z)}{\partial s} + \left[\frac{z+\lambda_0(1-B^*(s))-sa}{bs}\right]M^0(s,z) = \frac{M(s,0)-sa\Pi^0(z)}{bs}$$

Solving (7) we have

$$M^0(s,z)=\exp\{-\frac{z}{b}\int_1^s\frac{dx}{x} + \theta(s)\}\{c-\int_0^s \exp\left[\frac{z}{b}\int_1^x\frac{du}{u} -\theta(x)\right]\left[\frac{M(x,0)}{bx}-\frac{a}{b}\,\Pi^0(z)\right]dx\}$$

$$=s^{-z/b}e^{\theta(s)}\{c-\int_0^s x^{z/b}e^{-\theta(x)}\left[\frac{M(x,0)}{bx}-\frac{a}{b}\,\Pi^0(z)\right]dx\}$$

As $s\to\infty$, $s^{-z/b}e^{\theta(s)}\to\infty$ also. Since $M^0(s,z)\to\Pi^0(z)<\infty$ we must have

$$c = \int_0^\infty x^{z/b}\, e^{-\theta(x)}\left[\frac{M(x,0)}{bx} - \frac{a}{b}\,\Pi^0(z)\right] dx.$$

Thus

$$(8)\quad M^0(s,z)=s^{-z/b}\, e^{\theta(s)}\{\int_s^\infty x^{z/b}\, e^{-\theta(x)}\left[\frac{M(x,0)}{bx} - \frac{a}{b}\,\Pi^0(z)\right]dx\}$$

Evidently, $\lim_{s\to\infty} M^0(s,z)=z^{-1}\neq\infty$. But $\lim_{s\to\infty} s^{-z/b}\, e^{\theta(s)}=\infty$. Then we must have that the terms inside the braces in (8) go to zero. This yields the following expression for $\Pi^0(z)$:

$$(9)\quad \Pi^{o}(z) = \frac{\int_0^\infty x^{z/b-1}\, e^{-\theta(x)}\, M(x,o)dx}{a\int_0^\infty x^{z/b}\, e^{-\theta(x)}\, dx}$$

The value of $\Pi^{o}(z,\lambda_o,W_1)$ is dependent on the initial distribution $W_1(x)=\tilde{W}(x)*B(x)$. Thus we need to substitute $\tilde{M}(u)B*(s)$ for $M(x,o)=M(x,o,\lambda_o,W_1)$ in equation (9), yielding:

$$\Pi^{o}(z,\lambda_o,W_1)= \frac{\int_0^\infty x^{z/b-1}\, e^{-\theta(x)}\, e^{\theta(x)} \left[\int_x^\infty e^{-\theta(u)}du\right] / \left[\int_0^\infty e^{-\theta(u)}du\right] \; B^{*}(x)dx}{a\int_0^\infty e^{-\theta(x)}\, dx}$$

$$(10)\qquad \frac{\int_0^\infty x^{z/b-1}\, B*(x) \int_x^\infty e^{-\theta(u)}\, du\, dx}{= a\left[\int_0^\infty x^{z/b}\, e^{-\theta(x)}dx\right]\left[\int_0^\infty e^{-\theta(x)}dx\right]}$$

5.2 A Special Case: Computational Aspects of Π^{o}

We now have all the tools to evaluate $\tilde{\Pi}(\tau,\lambda(\cdot))$. All that is needed is to designate values for a, b, B(x) and $\phi(t)$. We will specialize here to the case $\phi(t)=\cos 2\pi t$, $a=1$, $b=\lambda_o$ and $B(x)=\beta_1 e^{-x/\beta_1}$. We assume that $1+b\beta_1<e$ to ensure that a limit exists. This is so, since

$\lambda_o EY(U)= \frac{\lambda_o}{b} E\left[\ln(1+bU)\right] \leq \frac{\lambda_o}{b} \ln(1+b\beta_1)<1$ by Jensen's inequality. We note that $\theta(x)=\frac{x}{b} - \frac{\lambda_o}{b} \ln(1+\beta_1 x)$. Then

$$\Pi(\infty,\lambda_o) = \frac{b}{\int_0^\infty e^{-\theta(x)}dx}$$

$$(11) \qquad = (1+\beta_1 b)^{-1}$$

Also

$$(12)\ \Pi^0(z,\lambda_0,W_1) = \frac{\int_0^\infty u^{z/b-1}(1+\beta_1 u)^{-1} \int_0^\infty e^{-x/b}(1+\beta_1 x)dxdu}{\left[\int_0^\infty u^{z/b}\ e^{-u/b}(1+\beta_1 u)du\right]\left[\int_0^\infty e^{-u/b}(1+\beta_1 u)du\right]}$$

Computationally, there are still a few considerations, as the evaluation of $\Pi^0(z)$ at $z= \pm\ 2\pi i$ can present difficulty. We determine the inner integral of the numerator in (12) as

$$\int_u^\infty (1+\beta_1 x)e^{-x/b}dx=(1+\beta_1 u)be^{-u/b}+b^2\beta_1 e^{-u/b}$$

Let N stand for the numerator in (12); then

$$N = b \int_0^\infty u^{z/b-1}\ e^{-u/b}du + b^2\beta_1 \int_0^\infty e^{-u/b}u^{z/b-1}(1+\beta_1 u)^{-1}du$$

$$= N_1 + N_2,\ \text{say.}$$

N_1 is easily evaluated as proportional to a gamma function, using the simple change of variable u=by; indeed

$$N_1 = b^{z/b+1}\int_0^\infty e^{-y}\ y^{z/b-1}\ dy$$

$$(13) \qquad = b^{z/b+1}\ \Gamma(z/b)$$

The integral $\Gamma(\pm \frac{2\pi i}{b})$ cannot readily be evaluated, but $\Gamma(\pm \frac{2\pi i}{b} + 1)$ can, and we then can use the identity $\Gamma(x+1)=x\Gamma(x)$. We have

$$\Gamma(\pm \frac{2\pi i}{b} + 1) = \int_0^\infty e^{-y} y^{\pm 2\pi i/b} dy$$

$$= \int_0^\infty e^{-y} e^{\pm 2\pi i/b \log y} dy$$

$$= \int_0^\infty e^{-y} \cos\left[\frac{2\pi}{b} \log y\right]dy \pm i \int_0^\infty e^{-y} \sin\left[\frac{2\pi}{b} \log y\right]dy$$

The two functions on the right are easily computable.

The evaluation of N_2 cannot be done straightforwardly. We write

$$N_2 = b^2\beta_1 \int_0^\infty e^{-u/b} u^{z/b-1}(1+\beta_1 u)^{-1} du$$

$$= b^2\beta_1 \int_0^\infty e^{-u/b} u^{z/b-1}\left[(1+\beta_1 u)^{-1}-1\right]du+b^2\beta_1 \int_0^\infty e^{-u/b} u^{z/b-1} du$$

$$(14) \quad = -b^2\beta_1{}^2 \int_0^\infty e^{-u/b} u^{z/b}(1+\beta_1 u)^{-1} du + b\beta_1 N_1$$

We note that the first integral on the right can be written, for $z=\pm 2\pi i$, as

$$- b^2\beta_1{}^2\int_0^\infty e^{-u/b} u^{\pm 2\pi i/b}(1+\beta_1 u)^{-1} du =$$

$$(15) \quad -b^2\beta_1{}^2\int_0^\infty e^{-u/b}\cos\left[\frac{2\pi i}{b} \log u\right](1+\beta_1 u)^{-1}du \mp b^2\beta_1{}^2\int_0^\infty e^{-u/b}\sin\left[\frac{2\pi i}{b}\log u\right](1+\beta_1 u)^{-1}du.$$

If we also write D for the denominator in (12) we have

$$D = \left[\int_o^\infty (1+\beta_1 x)e^{-x/b}\, x^{z/b} dx\right]\left[\int_o^\infty e^{-u/b}(1+\beta_1 u)du\right]$$

$$= \frac{b}{\Pi(\infty,\lambda_o)} \left[\int_o^\infty e^{-x/b}\, x^{z/b} dx + \beta_1 \int_o^\infty e^{-x/b}\, x^{z/b+1}\, dx\right]$$

$$= \frac{b}{\Pi(\infty,\lambda_o)}\left[b^{z/b+1}\Gamma(\tfrac{z}{b}+1) + \beta_1 b^{z/b+2}\Gamma(\tfrac{z}{b}+2)\right]$$

$$(16) \quad = \frac{b^{z/b+2}}{\Pi(\infty,\lambda_o)} \left[1+\beta_1 z+\beta_1 b\right]\Gamma(\tfrac{z}{b}+1)$$

With the above representations of N_1, N_2 and D we can compute $\Pi^o(z)$ and thus $\tilde{\Pi}(\tau,\lambda(\cdot))$.

5.3 Limiting Value of the Mean Store Content; r(x)=a+bx

The mean of the storage process is also of particular importance. To derive the approximation we must determine the mean for the homogeneous store. Clearly, from (7)

$$\lim_{s\to\infty} \frac{\partial M^o(s,z)}{\partial s} = -\frac{a}{b}\, \Pi^o(z) + \frac{a}{b} \lim_{s\to\infty} M^o(s,z) + \lim_{s\to\infty}\left[\frac{M(s,o)-(z+\lambda_o(1-B^*(s)))M(s,z)}{bs}\right]$$

We note:

(1) $\lim_{s\to\infty} M^o(s,z) = \frac{1}{z}$;

(2) By l'Hôpital's rule

$$\lim_{s\to\infty} \frac{M(s,o)-(z+\lambda_o(1-B^*(s)))M(s,z)}{bs} = \frac{-\mu(0)+z\mu^o(z)-\lambda_o\beta_1/z}{b}$$

This yields

$$\mu^o(z)(z+b) = \mu(0) + (\lambda_o\beta_1 - a)/z + a\Pi^o(z)$$

which implies

$$\mu(t) = \mu(0)e^{-bt}+(\lambda_o\beta_1-a)\int_o^t e^{-bu}du+a\int_o^t \Pi(u)e^{-b(t-u)}du$$

$$(17)\qquad = \left(\mu(0)-\frac{\lambda_o\beta_1}{b}+\frac{a}{b}\right)e^{-bt}+\frac{\lambda_o\beta_1-a}{b}+a\int_o^t \Pi(u)e^{-b(t-u)}du.$$

Having performed this preliminary calculation of $\mu(t)$, we substitute (17) into equation 3.17 to obtain the desired approximation. Note that $\mu(0,\lambda_o,W_1) = \mu(\infty,\lambda_o) + \beta_1$. Thus

$$\mu(t,\lambda(\cdot),W)=\mu(\infty,\lambda_o)(1-\varepsilon\lambda_o\Phi(t))+\varepsilon\lambda_o\int_o^t \phi(t-u)\mu(u,\lambda_o,W_1)du+O(\varepsilon^2)$$

$$=\mu(\infty,\lambda_o)-\varepsilon\lambda_o\mu(\infty,\lambda_o)\Phi(t)+\varepsilon\lambda_o\Bigg[\frac{\lambda_o\beta_1-a}{b}\int_o^t \phi(u)du$$

$$+\left(\mu(\infty,\lambda_o)+\beta_1-\frac{\lambda_o\beta_1-a}{b}\right)\int_o^t \phi(t-u)e^{-bu}du$$

$$+a\int_o^t \phi(t-u)\int_o^u \Pi(x,\lambda_o,W_1)\, e^{-b(u-x)}dxdu\Bigg]+O(\varepsilon^2)$$

$$=\frac{1}{b}(a\Pi(\infty,\lambda_o)-a+\lambda_o\beta_1)-\varepsilon\lambda_o\,\frac{a\Pi(\infty,\lambda_o)}{b}\Phi(t)$$

$$(18)\quad +\varepsilon\lambda_o\left[\frac{a}{b}\Pi(\infty,\lambda_o)+\beta_1\right]\int_o^t \phi(u)e^{-b(t-u)}du+\varepsilon\lambda_o a\int_o^t \phi(t-u)\int_o^u \Pi(x,\lambda_o,W_1)e^{-b(u-x)}dxdu+O(\varepsilon^2)$$

5.4 A Special Case: $\phi(u)=\cos 2\pi u$

In the case $\phi(u)=\cos 2\pi u$, $\Phi(t)=\frac{\sin(2\pi t)}{2\pi}$ and

$$\int_o^t \cos 2\pi u e^{-b(t-u)}du=e^{-bt}\left\{e^{bu}\frac{[b\cos 2\pi u+2\pi\sin 2\pi u]}{b^2+4\pi^2}\right\}_o^t$$

(19) $$= \frac{b\cos 2\pi t+2\pi\sin 2\pi t-be^{-bt}}{b^2+4\pi^2}$$

To determine $\tilde{\mu}(\tau,\lambda(\cdot))$ we need to study, from (18),

$\int_0^t \cos 2\pi(t-u) \int_0^u \Pi(x,\lambda_o,W_1)\ e^{-b(u-x)}dxdu = J(t)$, say. Since

$|\cos 2\pi(t-u)e^{-b(u-x)}\Pi(x,\lambda_o,W_1)| \leq e^{-b(u-x)}$, integrable in $[o,t]x[o,u]$, we

can interchange integrals to get

$$J(t) = \int_0^t \Pi(x,\lambda_o,W_1) \int_x^t \cos 2\pi(t-u)e^{-b(u-x)}dudx$$

$$= \int_0^t \Pi(x,\lambda_o,W_1) \left\{\frac{b\cos 2\pi(t-x)+2\pi\sin 2\pi(t-x)-be^{-b(t-x)}}{b^2+4\pi^2}\right\} dx$$

by (19). Making the substitution $\hat{\Pi}(x)=\Pi(x)-\Pi(\infty)$ we have

$$J(t)=\frac{\Pi(\infty,\lambda_o)}{b^2+4\pi^2}\left\{b\int_0^t \cos 2\pi udu+2\pi\int_0^t \sin 2\pi udu\right\} \frac{-b\int_0^t \Pi(x,\lambda_o,W_1)e^{-b(t-x)}\,dx}{b^2+4\pi^2}$$

$$+\frac{b}{b^2+4\pi^2}\int_0^t \hat{\Pi}(x,\lambda_o,W_1)\cos 2\pi(t-x)dx+\frac{2\pi}{b^2+4\pi^2}\int_0^t\hat{\Pi}(x,\lambda_o,W_1)\sin 2\pi(t-x)dx.$$

To determine $\tilde{\mu}(\tau,\lambda(\cdot))$ we need to determine $\lim_{n\to\infty} J(n+\tau)$. As we saw in the previous chapter

$$\lim_{n\to\infty} \int_0^{n+\tau} \hat{\Pi}(x,\lambda_o,W_1)\cos 2\pi(t-x)dx=\frac{1}{2}e^{2\pi i\tau}\hat{\Pi}^o(2\pi i,\lambda_o,W_1)+\frac{1}{2}e^{-2\pi i\tau}\hat{\Pi}^o(-2\pi i,\lambda_o,W_1)$$

and

$$\lim_{n\to\infty} \int_0^{n+\tau} \hat{\Pi}(x,\lambda_o,W_1)\sin 2\pi(t-x)dx=\frac{1}{2i}\ e^{2\pi i\tau}\Pi^o(2\pi i,\lambda_o,W_1)-\frac{1}{2i}\ e^{-2\pi i\tau}\hat{\Pi}^o(-2\pi i,\lambda_o,W_1)$$

From Lemma 3.2.2, since $\Pi(t,\lambda_o,W_1)$ is bounded and converges to a limit, and $e^{-bx}\epsilon L_1$, we have that

$$\lim_{n\to\infty}\int_0^{n+\tau}\Pi(x,\lambda_o,W_1)e^{-b(n+\tau-x)}dx=\Pi(\infty,\lambda_o)\int_0^{\infty}e^{-bx}dx=\frac{1}{b}\Pi(\infty,\lambda_o).$$

Thus

$$\tilde{\mu}(\tau,\lambda(\cdot))=\frac{1}{b}(a\Pi(\infty,\lambda_o)-a+\lambda_o\beta_1)-\epsilon\lambda_o a\Pi(\infty,\lambda_o)\frac{\sin 2\pi\tau}{2\pi b}$$

$$+\ \epsilon\lambda_o\left[\frac{a}{b}\Pi(\infty,\lambda_o)+\beta_1\right]\left[\frac{b\cos 2\pi\tau+2\pi\sin 2\pi\tau}{b^2+4\pi^2}\right]$$

$$+\ \frac{\epsilon\lambda_o a\Pi(\infty,\lambda_o)}{b^2+4\pi^2}\left[\frac{b\sin 2\pi\tau}{2\pi}-\cos 2\pi\tau+1\right]-\frac{\epsilon\lambda_o a\Pi(\infty,\lambda_o)}{b^2+4\pi^2}$$

$$+\ \frac{\epsilon\lambda_o a e^{2\pi i\tau}}{b^2+4\pi^2}\hat{\Pi}^o(2\pi i,\lambda_o,W_1)\left(\frac{b}{2}+\frac{2\pi}{2i}\right)$$

$$+\ \frac{\epsilon\lambda_o a}{b^2 4\pi^2}\ e^{-2\pi i\tau}\hat{\Pi}^o(-2\pi i,\lambda_o,W_1)\ \left(\frac{b}{2}-\frac{2\pi}{2i}\right)$$

$$=\frac{a\Pi(\infty,\lambda_o)-a+\lambda_o\beta_1}{b}+\frac{\epsilon\lambda_o\beta_1}{b^2+4\pi^2}\left[b\cos 2\pi\tau+2\pi\sin 2\pi\tau\right]$$

$$+\ \frac{\epsilon\lambda_o a}{2(b^2+4\pi^2)}\left[(b-2\pi i)e^{2\pi i\tau}\hat{\Pi}^o(2\pi i,\lambda_o,W_1)+(b+2\pi i)e^{-2\pi i\tau}\hat{\Pi}^o(-2\pi i,\lambda_o,W_1)\right]\tag{20}$$

Equation (20) can alternatively be written without using complex numbers. First we note that $\hat{\Pi}^o(z)=\Pi^o(z)-\Pi(\infty)/z$. If we let $\Pi^o(2\pi i,\lambda_o,W_1)=x+iy$, then after some algebraic manipulation we can write (20) as

$$\tilde{\mu}(\tau,\lambda(\cdot)) = \frac{a\Pi(\infty,\lambda_o)-a+\lambda_o\beta_1}{b} + \frac{\varepsilon\lambda_o}{b^2+4\pi^2}\left\{b\beta_1+a(bx+2\pi y)+a\Pi(\infty,\lambda_o)\right\}\cos 2\pi\tau$$

$$(21) \qquad + \frac{\varepsilon\lambda_o}{b^2+4\pi^2}\left\{2\pi\beta_1-a(by-2\pi x)-a\,\frac{b\Pi(\infty,\lambda_o)}{2\pi}\right\}\sin 2\pi\tau$$

As before, we need to establish that $\lim_{n\to\infty} \mu(n+\tau,\lambda(\cdot),\tilde{W})$ is the expectation of the limiting value of store content. We look at the homogeneous store with $r(x)=a+bx$ and $\lambda_1=\lambda_o(1+\varepsilon)\geq\lambda(t)$. We know from (6) that $\mu(\infty,\lambda_1)=\frac{1}{b}(a\Pi(\infty,\lambda_1)-a+\lambda_1\beta_1)$ as long as $a+bx\geq\lambda_o\beta_1$. We also see from (17) that

$$\lim_{t\to\infty} \mu(t,\lambda_1,W) = \frac{\lambda_1\beta_1-a}{b} + \frac{a}{b}\Pi(\infty,\lambda_1)$$

$$= \mu(\infty,\lambda_1)$$

Since we know that a limiting distribution of store content exists, $\lim_{t\to\infty} \mu(t,\lambda(\cdot),W)=\mu(\infty,\lambda_o)$, and all random variables are nonnegative, we know that the store content is uniformly integrable. In addition, since by Lemma 3.2.1 we have $\mu(t,\lambda(\cdot),W)\leq\mu(t,\lambda_1,W)$, the content of the nonhomogeneous process must be uniformly integrable, which gives us what we seek--that for each $\tau,\tilde{\mu}(\tau,\lambda(\cdot))$ is the approximation to the expectation of the quasi-limiting value of store content.

5.5 Some Graphs

Figures 5.1-5.6 are examples of graphs of the probability of an empty

store and mean store content of the nonhomogeneous store with $r(x)=1+bx$, $\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$ and $\lambda_o=b$. We have $\beta_1=.1$ for all figures and $\lambda_o=1$, 5, 9.5. In each figure, ε takes on the four values .1, .25, .5 and .9.

The probability of an empty store when $r(x)=1+bx$ has form similar to the earlier examples. The mean store content, while similar in form, varies considerably more than other stores. In fact, at some values of τ and ε, $\tilde{\mu}(\tau,\lambda(\cdot))$ is less than 0. This is possible, since $\tilde{\mu}(\tau,\lambda(\cdot))$ is only an approximation, and many of the remaining terms in the series expansion of $\mu(\tau,\lambda(\cdot))$ may be positive. Of course, if $\tilde{\mu}(\tau,\lambda(\cdot))<0$ we take it to be zero.

Tables 5.1 and 5.2 give the extreme values, percentage deviation of the extrema from the average, and response time for $\hat{\Pi}(\tau,\lambda(\cdot))$ and $\tilde{\mu}(\tau,\lambda(\cdot))$ respectively. While not so indicated, the values of $\tilde{\mu}$ are assumed to be nonnegative. It should be noted that the difference in the figures reflect not only the differences in λ_o, but also the release rules, since we have assumed that $\lambda_o=b$. Thus we cannot directly compare either the graphs or the entries in the table.

Table 5.1

λ_o	Extreme Values	% Deviation from $\Pi(\infty,\lambda_o)$	Response Time
1	$.91\pm.08\varepsilon$	9ε	.0775
5	$.67\mp.27\varepsilon$	41ε	.0777
9.5	$.51\mp.37\varepsilon$	72ε	.0659

Table 5.2

λ_o	Extreme Values	% Deviation from $\mu(\infty,\lambda_o)$	Response Time
1	$.009\pm.007\varepsilon$	78ε	.0399
5	$.033\pm.044\varepsilon$	133ε	.4497
9.5	$.049\pm.038\varepsilon$	78ε	.4249

5.6 A Type B Process

Up to now we have spent a good deal of time examining the emptiness of a store. But there are some stores which cannot empty in a finite amount of time. These are Type B processes. The content of a Type B process can get arbitrarily close to zero, but $\Pi(t,\lambda(\cdot),W)\equiv 0$. What is of interest here is the mean store content. A typical release rule for a Type B process is $r(x)=cx$. For this particular store we will see that we can get more than just an approximation to the mean content.

We begin with the basic storage equation

$$Z(t)=Z(0) + A(t) - c \int_0^t Z(u)du$$

where $A(t)$ is the total input in $(0,t]$. Assuming the input is Compound Poisson, $\lambda(t)=\lambda_o(1+\varepsilon\phi(t))$, we take expectations of both sides of (22) to get

$$\mu(t,\lambda(\cdot),W)=\mu(0,\lambda(\cdot),W)+\beta_1 N(t)-c \int_0^t \mu(x,\lambda(\cdot),W)dx$$

(23) $$=\mu(0,\lambda(\cdot),W)+\beta_1(\lambda_o+\varepsilon\lambda_o\Phi(t))-c \int_0^t \mu(x,\lambda(\cdot),W)dx$$

We take Laplace-Transforms with respect to t, of both sides of (23) to get

$$\mu^o(s,\lambda(\cdot),W)=\frac{\mu(0,\lambda(\cdot),W)}{s}+\frac{\beta_1\lambda_o}{s}+\varepsilon\beta_1\lambda_o\Phi^o(s)-c\frac{\mu^o(s,\lambda(\cdot),W)}{s}$$

FIGURE 5.1

Probability of an Empty Store

$r(x)=1+x$

$\lambda_o=1.0 \ \beta_1=.10$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$

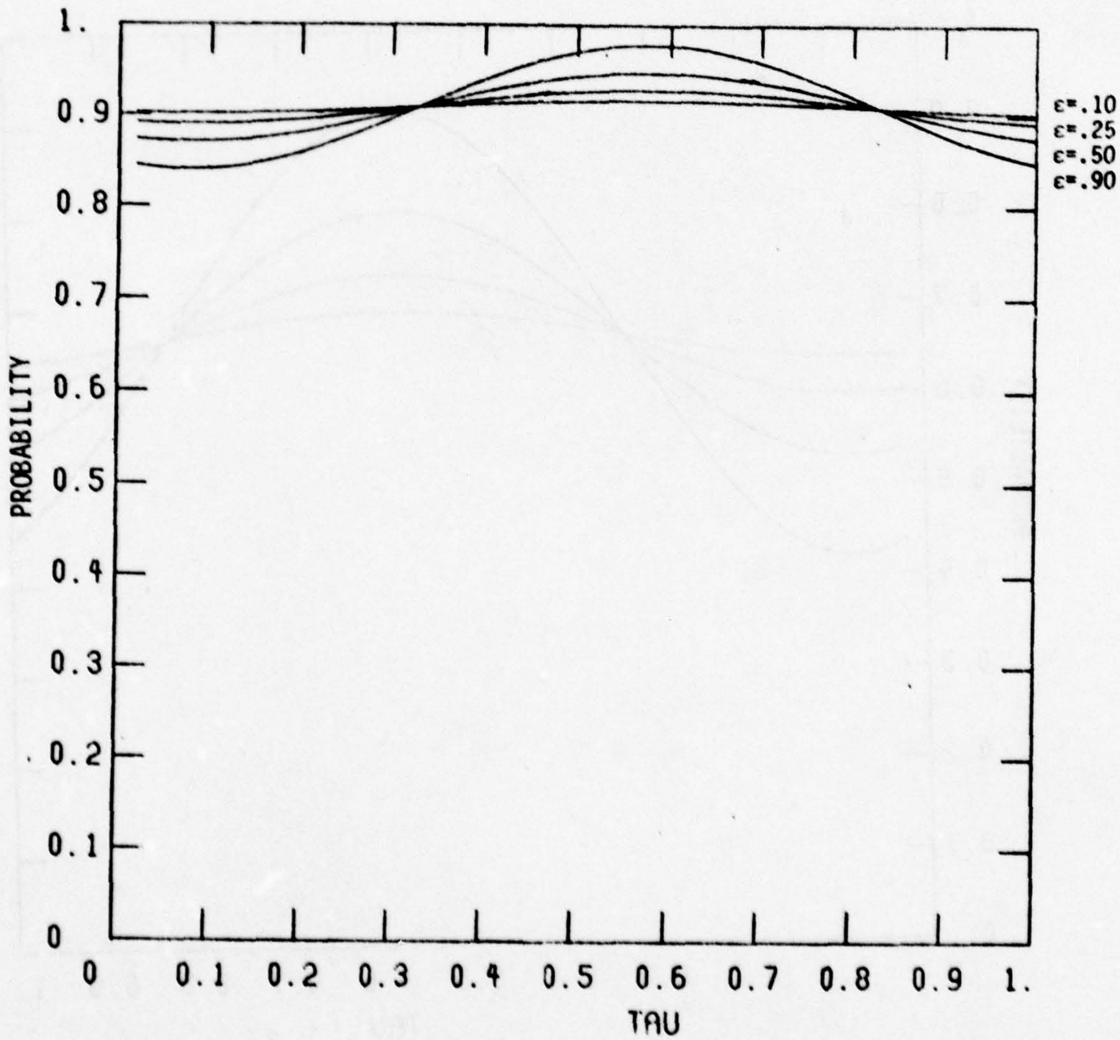


FIGURE 5.2

Probability of an Empty Store

$r(x)=1+5x$

$\lambda_o=5.0 \quad \beta_1=.10$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$

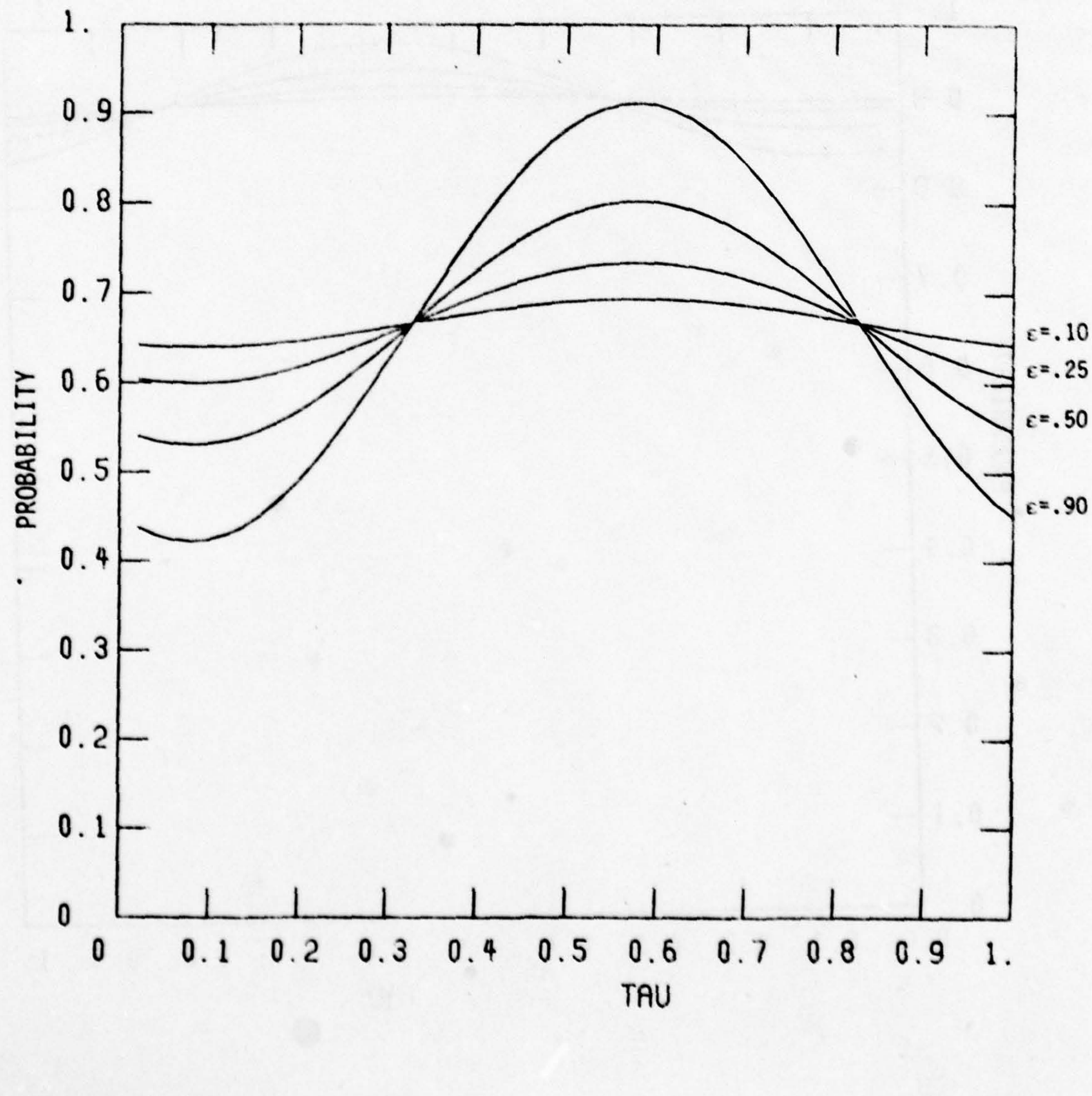


FIGURE 5.3

Probability of an Empty Store

$r(x)=1+9.5x$

$\lambda_o=9.5 \quad \beta_1=.10$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$

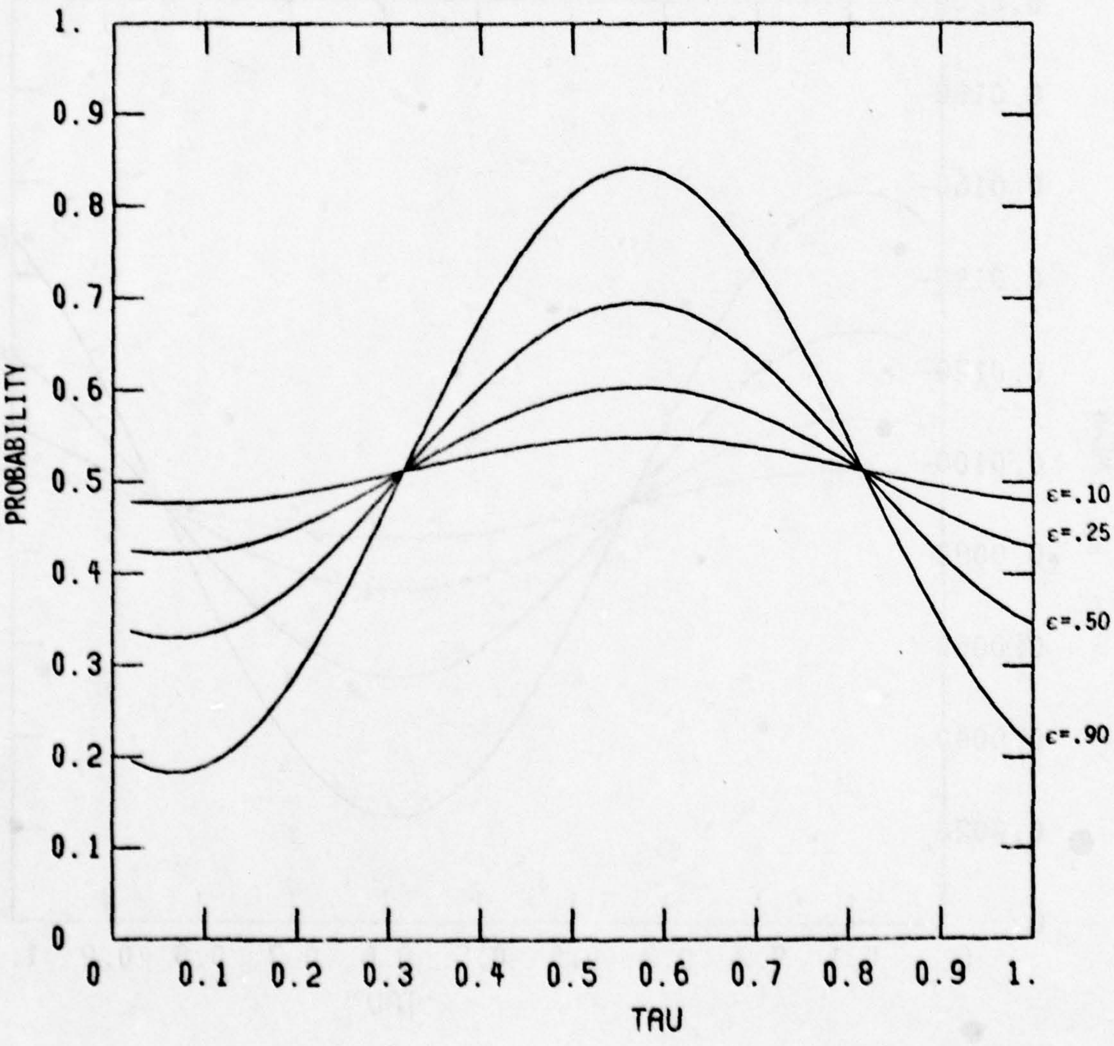


FIGURE 5.4

Mean Store Content

$r(x)=1+x$

$\lambda_o=1.0 \quad \beta_1=.10$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$

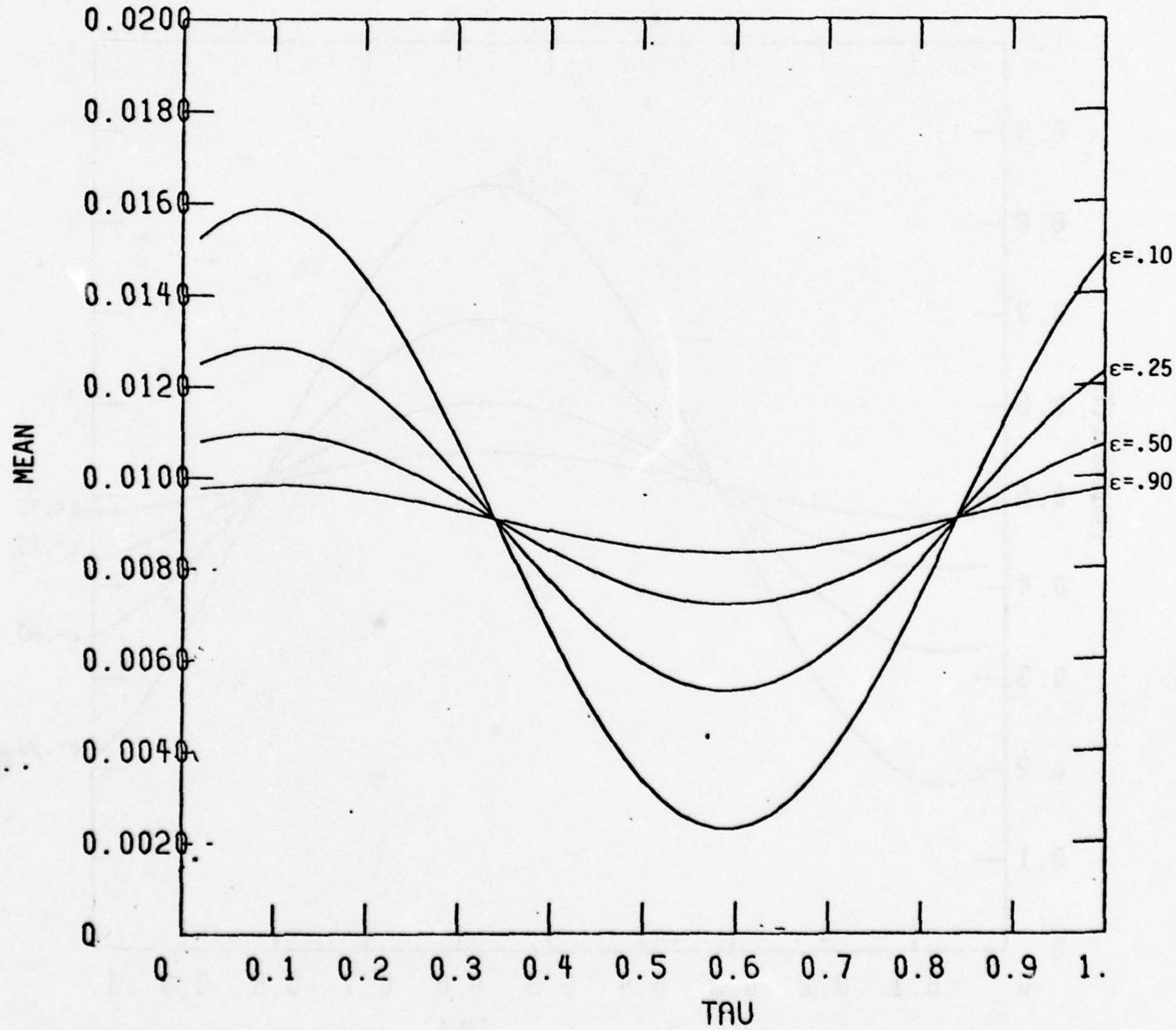


FIGURE 5.5

Mean Store Content

$r(x)=1+5x$

$\lambda_o=5.0 \quad \beta_1=.10$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$

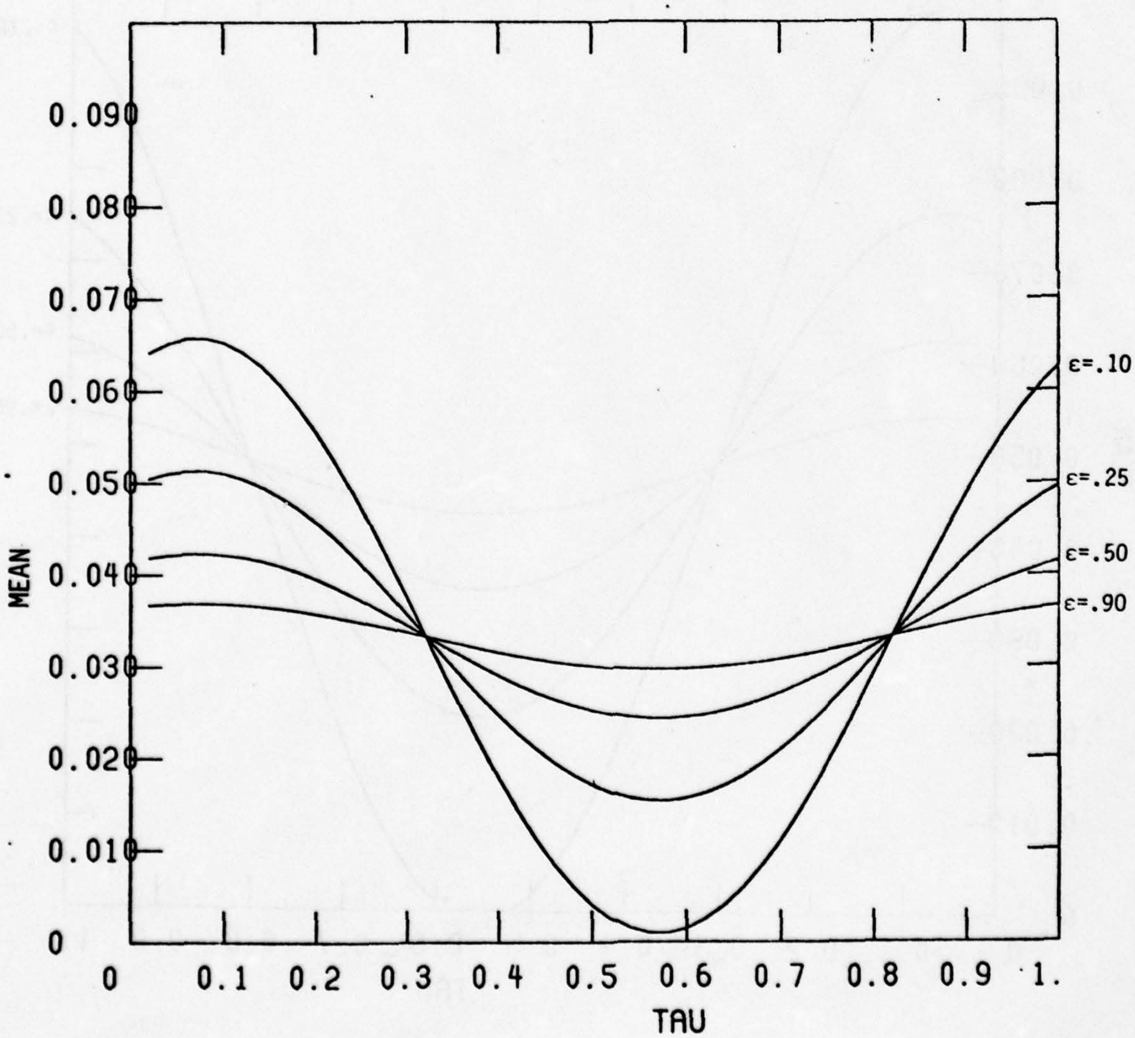


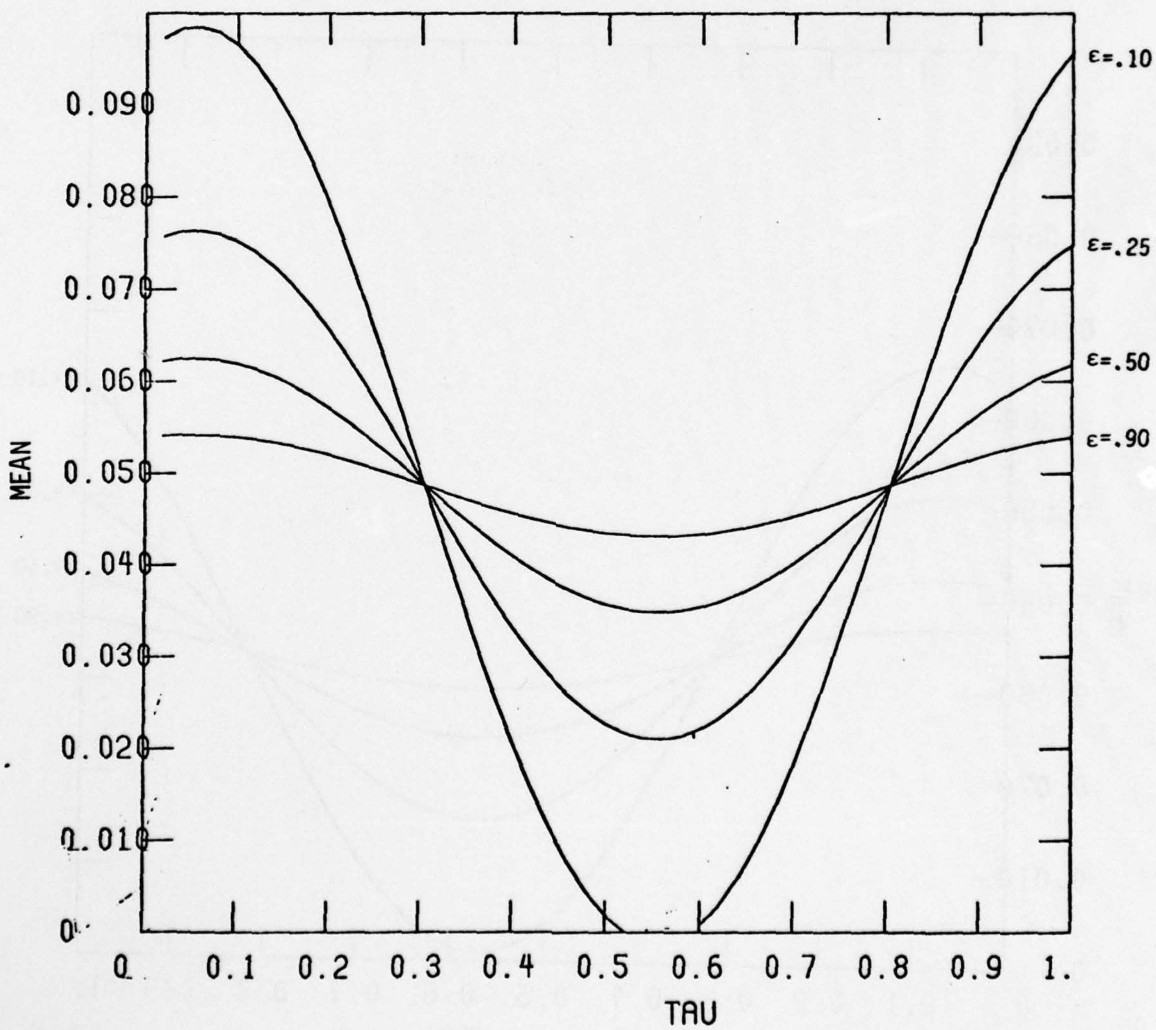
FIGURE 5.6

Mean Store Content

$r(x)=1+9.5x$

$\lambda_o=9.5 \quad \beta_1=.10$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$



Then

$$\mu(t,\lambda(\cdot),W) = \mu(o,\lambda(\cdot),W)e^{-ct}+\beta_1\lambda_o \int_o^t e^{-cu}du+\varepsilon\lambda_o\beta_1 \int_o^t \phi(u)e^{-c(t-u)}du$$

$$(24) \qquad = \left[\mu(o,\lambda(\cdot),W)-1\right]e^{-ct}+\frac{\beta_1\lambda_o}{c} +\varepsilon\lambda_o\,{}_1 e^{-ct}\int_o^t \phi(u)e^{cu}du$$

We take note here of a few important points. First, equation (24) only requires that $\int_o^t \phi(u)e^{-c(t-u)}du$ exist for all t. Second, we have shown that the entire expansion of μ in ε is contained in the first two terms. We have no need to rely on an approximation. Third, we see that the mean store content depends on the input size distribution only through the mean of the distribution.

We now restrict ourselves to the case of periodic $\phi(t)$. The last term in (24) becomes

$$\varepsilon\lambda_o\beta_1 e^{-ct}\int_o^t \phi(u)e^{cu}du = \varepsilon\lambda_o\beta_1 e^{-ct} \sum_{k=-M}^{M} a_k \int_o^t e^{2\pi iku}e^{cu}du$$

$$= \varepsilon\lambda_o\beta_1\left[\sum_{k=-M}^{M} a_k\frac{e^{2\pi ikt}}{2\pi ik+c} - e^{-ct} \sum_{k=-M}^{M} \frac{a_k}{2\pi ik+c}\right]$$

If we go to the limit as n goes to infinity we get

$$(25) \quad \mu(\tau,\lambda(\cdot))=\lim_{n\to\infty} \mu(n+\tau,\lambda(\cdot),W)=\frac{\beta_1\lambda_o}{c} + \varepsilon\lambda_o\beta_1 \sum_{k=-M}^{M} a_k\frac{e^{2\pi ik\tau}}{(2\pi ik+c)}$$

We would like to be able to say that $\mu(\tau,\lambda(\cdot))$ does indeed equal the mean of the quasi-limiting distribution of store content. First, we show

that a quasi-limiting distribution exists, as long as $\lambda_o\beta_1<c$, by virtue of Theorem 2.4. For

$$Y(U,\varepsilon) = \int_{\varepsilon}^{U} \frac{dx}{cx} = \frac{1}{c}\left[\ln U-\ln\varepsilon\right],$$

and

$$\mu_\varepsilon = \left[\int_0^1 \lambda(u)du\right] EY(U,\varepsilon)$$

$$= \lambda_o \frac{1}{c}\left[E\ln U-\ln\varepsilon\right].$$

The function $f(x)=\ln x$ is concave, so by Jensen's inequality

$$E\ln U \leq \ln\left[EU\right] \leq E\ U = \beta_1$$

Thus, for any $\varepsilon>0$

$$\mu_\varepsilon \leq \frac{1}{c}\lambda_o\left[\beta_1-\ln\varepsilon\right] \leq \frac{1}{c}\lambda_o\beta_1 < 1,$$

and the assumptions of Theorem 2.4 are satisfied.

Let us now consider the homogeneous store with $\lambda_1=\lambda_o(1+\varepsilon M)$, where $\phi(t)\leq M$. The Laplace-Stieltjes Transform of store content can be determined from the partial differential equation

$$(26)\quad \frac{\partial M(s,t,\lambda_1,W)}{\partial t}+\lambda_1(1-B^*(s))M(s,t,\lambda_1,W)= -\frac{sc\partial M(s,t,\lambda_1,W)}{\partial s}$$

Let $M^*(s,\lambda_1)$ denote the Laplace Stieltjes Transform of the limiting value of store content. We know this exists, since $r(x)>\lambda_1\beta_1$ for $x>\lambda_1\beta_1$, so Theorem 2.5 applies. Then $M^*(s,\lambda_1)$ satisfies

(27) $\frac{d\,M^*(s,\lambda_1)}{ds} = \frac{-\lambda_1(1-B^*(s))M^*(s,\lambda_1)}{cs}$

To find $\mu(\infty,\lambda_1)$ we take the limit of $\frac{dM^*}{ds}$ as $s\to 0$, in (27). We get $\mu(\infty,\lambda_1) = \lambda\beta_1/c$.

But we already know that

$$\mu(t,\lambda_1,W) = \mu(0,\lambda_1,W)\, e^{-ct} + \frac{\beta_1\lambda_1}{c}(1-e^{-ct})$$

so

$$\lim_{t\to o} \mu(t,\lambda_1,W) = \mu(\infty,\lambda_1) = \frac{\lambda_1\beta_1}{c}$$

Since the store content and limiting value of store content in the homogeneous store are both nonnegative random variables, we have that the content is uniformly integrable. By Lemma 3.2.2, since $\lambda(t) \leq \lambda_1$ we have that $\mu(t,\lambda(\cdot),W) \leq \mu(t,\lambda_1,W)$, from which we can infer that the content in the nonhomogeneous store is uniformly integrable. With this we can conclude that for the nonhomogeneous store

$$\lim_{n\to\infty} E\left[Z(n+\tau)\right] = E\left[\lim_{n\to\infty} Z(n+\tau)\right]$$

5.7 An Example: $\phi(t)=\cos 2\pi t$

When $\phi(t)=\cos 2\pi t$ equation (25) becomes

$$\mu(\tau,\lambda(\cdot)) = \frac{\beta_1\lambda_o}{c} + \frac{\varepsilon\lambda_o\beta_1}{c}\left[\frac{1}{2}\,\frac{e^{-2\pi i}}{c-2\pi i} + \frac{1}{2}\,\frac{e^{2\pi i\tau}}{c+2\pi i}\right]$$

$$= \frac{\beta_1\lambda_o}{c} + \frac{\varepsilon\lambda_o\beta_1}{c^2+4\pi^2} \; [c \cos 2\pi\tau + 2\pi \sin 2\pi\tau]$$

We now look at a specific example, the store with $r(x) = \frac{1}{3}x$, $\lambda(t) = \lambda_o(1+\varepsilon\cos 2\pi t)$. Figures 5.7-5.9 are graphs of the mean content for this store. We let $\beta_1=.1$ and $\lambda_o=1, 5, 9.5$ for the three figures, while ε takes on the values .1, .25, .5 and .9 in each figure.

The graphs of the mean store content are symmetric about $\mu(\infty,\lambda_o)$. The extrema occur at points τ_o such that

$$\tan 2\pi\tau_o = \frac{2\pi}{c} .$$

We note that τ_o does not depend on either the input rate or input size distribution. Thus the response time is only a function of c. The actual value of the extrema points, though, do depend on the mean input size and the average intensity λ_o.

Table 5.3 gives the extrema, percent deviation of extrema from $\mu(\infty,\lambda_o)$ and the response time for each graph. The response time, of course, is the same for all λ_o, while the percent deviations of the extrema from $\mu(\infty,\lambda_o)$ remain virtually constant. Thus, except for a scale factor, for a given ε the graphs are essentially identical.

Table 5.3

λ_o	Extreme Values	% Deviation from $\mu(\infty,\lambda_o)$	Response Time
1	$.3 \pm .02\varepsilon$	6.6ε	.2416
5	$1.5 \mp .1\varepsilon$	6.6ε	.2416
9.5	$2.85 \mp .2\varepsilon$	7ε	.2416

FIGURE 5.7

Mean Store Content

$r(x)=\frac{1}{3}x$

$\lambda_0=1.0 \quad \beta_1=.10$

$\lambda(t)=\lambda_0(1+\varepsilon\cos 2\pi t)$

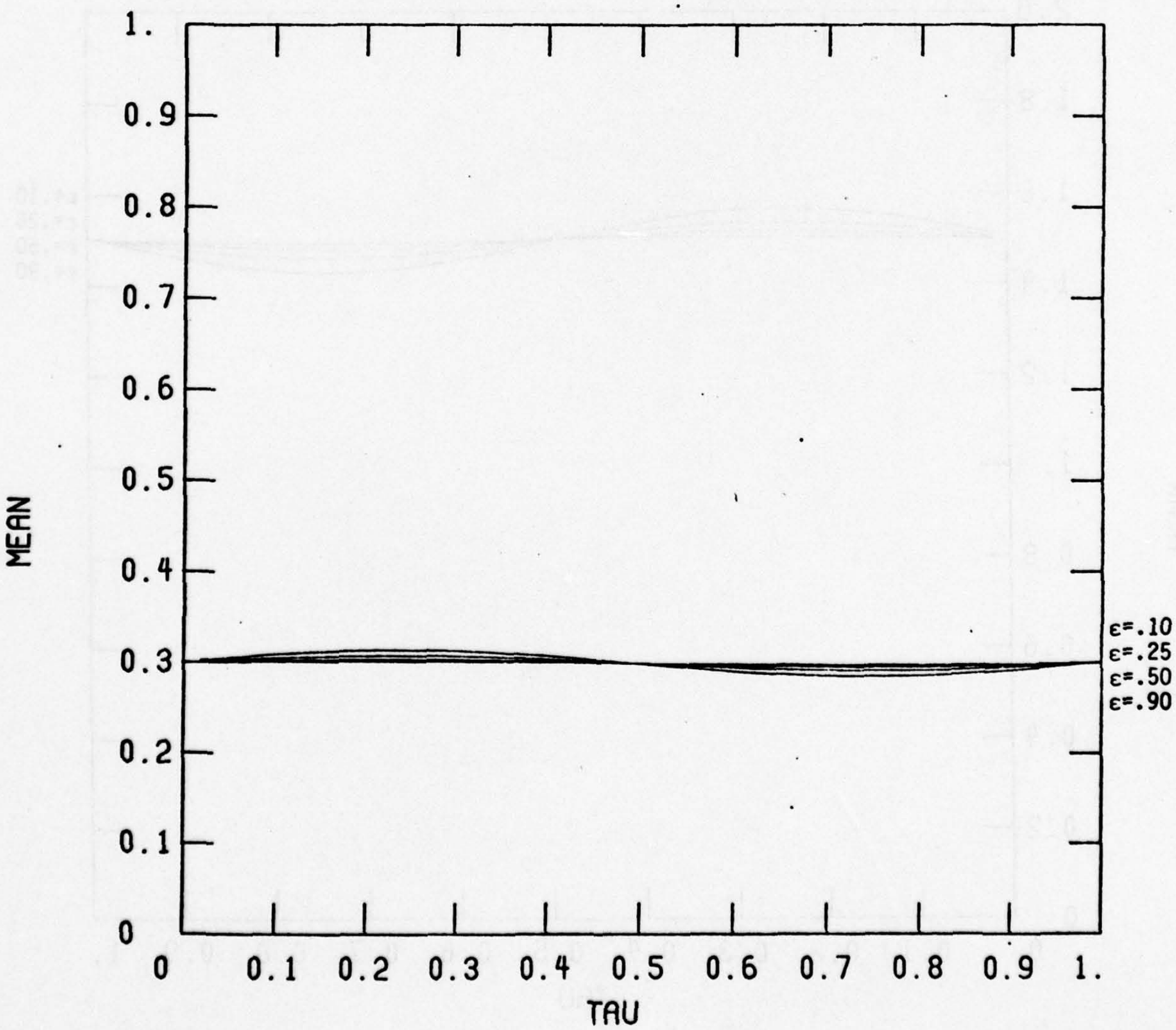


FIGURE 5.8

Mean Store Content

$r(x)=\frac{1}{3}x$

$\lambda_0=5.0 \quad \beta_1=.10$

$\lambda(t)=\lambda_0(1+\varepsilon\cos 2\pi t)$

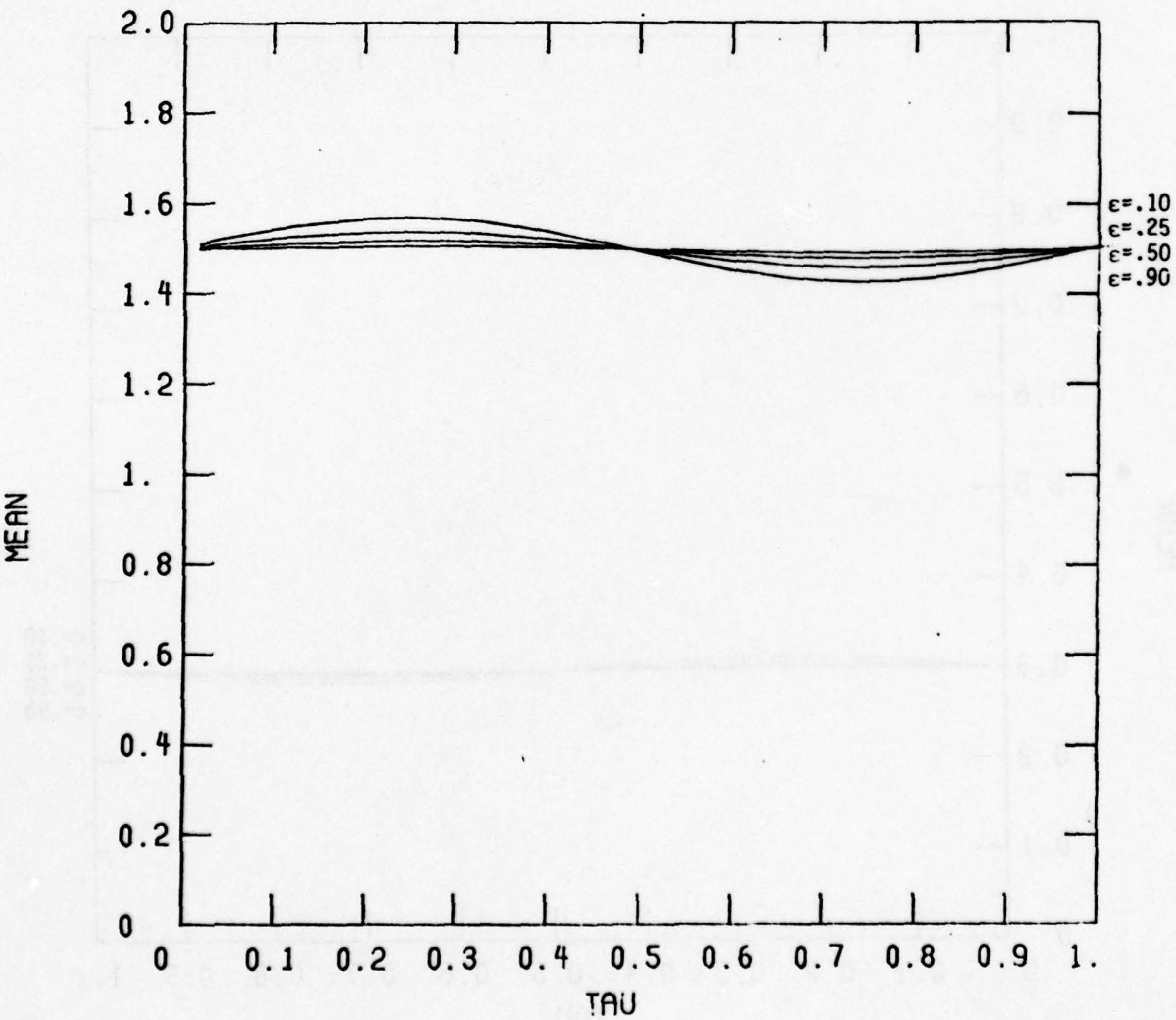


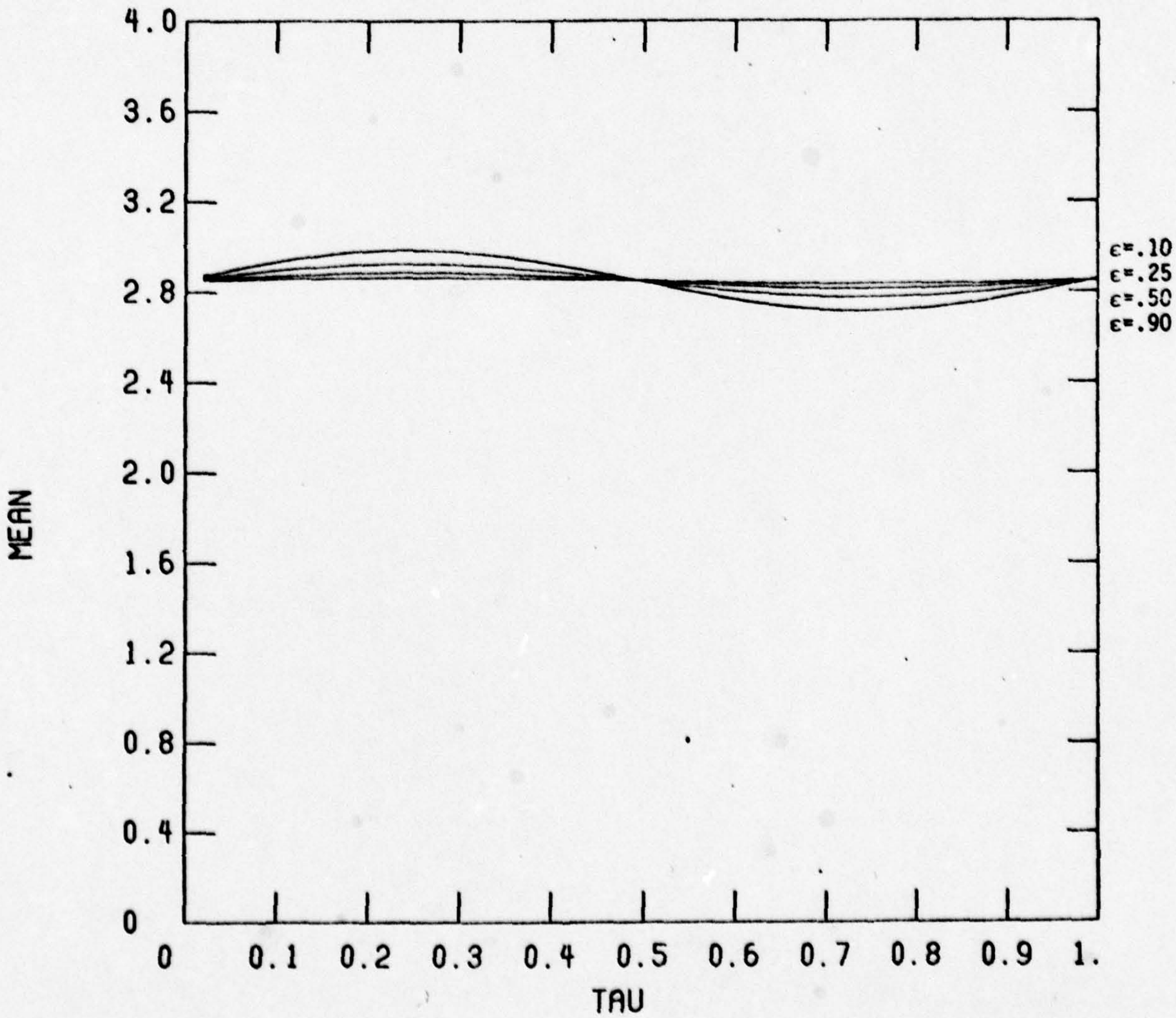
FIGURE 5.9

Mean Store Content

$r(x)=\frac{1}{3}x$

$\lambda_o=9.5 \quad \beta_1=.10$

$\lambda(t)=\lambda_o(1+\varepsilon\cos 2\pi t)$



CHAPTER 6

A MATRIX APPROACH TO THE NONHOMOGENEOUS QUEUE

6.1 A Step Function Approximation to $\lambda(t)$

In this chapter we depart from our previous perturbation approach to the nonhomogeneous store and look at an alternative method for dealing with the M(t)/M/1 queue which relies heavily on the computer. The method was discussed in a paper by Leese (1967). Basically, the method involves using a step function approximation to the arrival intensity to generate transition matrices for each time period during which the arrival intensity is constant. More specifically, suppose we assume that the nonnegative real line $[o,\infty)$ is divided into intervals $I_n=[t_{n-1},t_n)$ where $t_o=0$, $\bigcup_{k=1}^{\infty} I_k=[o,\infty)$, and on each interval the arrival intensity is approximated by a constant, i.e. $\lambda(t)\simeq\lambda_k$ for $t\varepsilon I_k$. If this is done, then on each interval I_k we essentially have an M/M/1 queue with arrival intensity λ_k.

We can compute the probability transition matrix $\underline{P}^k(t)$, which has entries

$$P^k_{N,n}(t) = P\{Q^k(t) = n \mid Q^k(0) = N\},\ t\varepsilon I_k$$

where $Q^k(t)$ is the number of customers in the system at time $t\varepsilon I_k$, using the well known formula for the transient probability:

$$P^k_{N,n}(t) = e^{-(\lambda_k+\mu)t}\left[\left(\frac{\mu}{\lambda_k}\right)^{(N-n)/2} (2\sqrt{\lambda_k\mu}\ t)\right.$$

$$(1) \qquad + \left(\frac{\mu}{\lambda_k}\right)^{(N-n+1)/2} I_{n+N+1}\left(2\sqrt{\lambda_k \mu}\, t\right)$$

$$+ \left(1 - \frac{\lambda_k}{\mu}\right)\left(\frac{\lambda_k}{\mu}\right)^n \sum_{l=n+N+2}^{\infty} \left(\frac{\mu}{\lambda_k}\right)^{l/2} I_l \left(2\sqrt{\lambda_k \mu}\, t\right)\Bigg]$$

Since the system is Markovian, by repeated multiplication of the $\underline{P}^k$ we can find the system size probabilities. For example, suppose $t \varepsilon I_2$ and we want to know the probability there are n in the system at time t if we started at time 0 with N. Looking at t_1 we see that there might have been j customers in the system at that time with probability $P^1_{N,j}(t_1)$. The probability of going from j to n in the system in the remaining $t-t_1$ time is $P^2_{j,n}(t-t_1)$. This gives

$$P_{N,n}(t) = \sum_{j=o}^{\infty} P^1_{N,j}(t_1) P^2_{j,n}(t-t_1)$$

This is just the (N,n) entry in the matrix product $\underline{P}^1(t_1)\underline{P}^2(t-t_1)$. It follows in general that if $t \varepsilon I_k$ then $P\{Q(t)=n \mid Q(0)=N\}$ is just the (N,n) entry in the matrix product.

$$(2) \qquad \underline{P}^1(t_1)\underline{P}^2(t_2-t_1). \, . \, .\underline{P}^{k-1}(t_{k-1}-t_{k-2})\underline{P}^k(t-t_{k-1})$$

In his paper, Leese complained that even with a judicious choice of step function and form of the formula to compute (1), the amount of computation required would be formidable and perhaps excessive. Thus he did not recommend this technique as a practical approach to the non-homogeneous queue. Since Leese wrote his paper more than 10 years ago,

much progress has been made in computer technology. It is likely that in many instances the amount of computation required would now be considered routine.

In the particular case of periodic arrival intensity, this matrix method seems well suited, even perhaps in Leese's time. For periodic $\lambda(t)$ the step function need only be fitted to the first period, $I=[o,\tilde{\omega})$. If $I=\bigcup_{k=1}^{n} I_1$ and $\lambda(t)\equiv\lambda_i$ for $t\varepsilon I_i$ then we only need N matrices to find $\underline{P}(\tilde{\omega})$. To find the limiting value of the size of the system, we continue squaring $\underline{P}=\underline{P}(\tilde{\omega})$ until the limit is reached. This should not require many multiplications, as we have:

$$
\begin{aligned}
&\underline{P}(2\tilde{\omega}) = \underline{P} \cdot \underline{P} = \underline{P}^2 \\
(3) \qquad &\underline{P}(4\tilde{\omega}) = \underline{P}(2\tilde{\omega}) \cdot \underline{P}(2\tilde{\omega}) = \underline{P}^4 \\
&\vdots \\
&\underline{P}(2^k\tilde{\omega}) = \underline{P}(2^{k-1}\tilde{\omega}) \cdot \underline{P}(2^{k-1}\tilde{\omega}) = \underline{P}^{2^k}
\end{aligned}
$$

Thus to get $\underline{P}(2^k\tilde{\omega})$ required k multiplications once $\underline{P}(\tilde{\omega})$ is obtained.

6.2 Construction of the Transition Matrices

Let us look at this method in greater detail. We shall first consider the construction of the basic transition matrix $\underline{P}^k(t)$. The formula for the entries in this matrix was given by (1). This form is not well suited for numerical work, as it is not clear how many terms of the infinite series of Bessel functions are needed for any particular level of accuracy. Leese suggests rewriting the formula as a power series in λ, but leaving the factor $e^{-(1+\lambda)t}$. This gives the much simpler formula:

$$(4)\quad P_{N,n}(t) = \sum_{m=o}^{\infty} \left[\frac{(\lambda t)^m e^{-\lambda t}}{m!}\right] F^m_{N,n}(t)$$

where

$$F^m_{N,n}(t) = S_{m+N-n}(t) + \frac{T_{m-n}(t)-T_{m-n-1}(t)}{T_m(t)} \cdot M(m+N+1,t)$$

$$T_j(t) = \begin{cases} \frac{(\mu t)j}{j!} & j\underline{>}0 \\ 0 & j>0 \end{cases}$$

(5)

$$S_j(t) = e^{-t\mu}T_j(t)$$

$$M(j,t) = \sum_{u=j}^{\infty} S_u(t) = 1- \sum_{u=o}^{j-1} S_u(t) .$$

It is interesting and helpful to note that the terms in (4) can be interpreted probabilistically. The factor $\frac{(\lambda t)^m e^{-\lambda t}}{m!}$ is the probability of exactly m arrivals in [0,t], while the factor $F^m_{N,n}(t)$ is the probability of there being n in the system at time t, given that there were N in the system at time 0, and there were exactly m arrivals in [0,t]. Since F is a probability we can make use of the fact that it is bounded above by 1 to determine the number of terms we need to use in (4). We have that

$$P_{N,N}(t) \leq \sum_{m=o}^{k} \frac{(\lambda t)^m e^{-\lambda t}}{m!} F^m_{N,n}(t) + \sum_{m=k+1}^{\infty} \frac{(\lambda t)^m e^{-\lambda t}}{m!} .$$

The error term

$$\sum_{m=k+1}^{\infty} \frac{(\lambda t)^m e^{-\lambda t}}{m!} = 1-e^{-\lambda t} \sum_{m=o}^{k} \frac{(\lambda t)^m}{m!}$$

can easily be calculated, and an appropriate value of k chosen for any given level of accuracy. Another nice aspect of (4) is that, if we can choose a step function with equal-sized time intervals then only one set of F values is needed.

The transition matrix P̲ is, theoretically, doubly infinite. Of course, computationally that is not feasible. Leese did not deal with this particular problem in the method. Naturally, we must use a finite matrix. What we have done here is to truncate the matrix, making it nxn. To retain the stochastic nature of the matrix, we replace the last entry in each row by $1 - \sum_{j=o}^{n-2} P_{ij}(t)$. Thus the last entry is in a sense the probability that, starting with i customers, the queue will have n-1 or more customers at time t. Since we also limit the number of rows, we are actually restricting the system to be no bigger than n-1, and hence the value of Π will be somewhat overestimated, and μ underestimated.

At what size should P̲ be truncated? This is a difficult question to answer. Here we only begin to explore the answer. Since we are essentially limiting the system size by truncating, we should look at the loss involved. For the homogeneous queue in steady state $P\{Q \geq n\} = \rho^n$ where $\rho = \lambda\beta_1$ is the traffic intensity. If, say, $\rho = .95$ then $P\{Q \geq 45\} = .099$; i.e., if we use a 45x45 transition matrix, allowing at most 44 customers in the system, we "turn away" or ignore 1 in 10 customers. Even if n=100, $P\{Q \geq 100\} = .006$. It seems that in some cases the

size of the matrix required to give reasonable results will surpass the limits of many computers. Consider further that to model a nonhomogeneous intensity will in general require a number of these matrices. The following table shows the "loss" involved in truncating a queue at size n-1, for a few values of ρ and n. It can be used as an aid in determining the size of matrix needed to be within a particular level of accuracy.

Table 6.1

$\rho \backslash n$	10	20	40	50	75	100	140
.95	.599	.358	.129	.077	.021	.066	<.001
.90	.349	.122	.015	.005	<.001		
.75	.056	.003	<.001				
.50	.001	<.001					
.30	<.001						

6.3 An Example: $\lambda(t)$ a Step Function

It is interesting to note an additional use of the matrix approach. For an M(t)/M/1 queue where $\lambda(t)$ is a step function, the matrix approach should yield values of Π and μ which are very nearly exact. If, in addition, $\lambda(t)$ is a periodic function, we can compare the values of $\Pi(\tau,\lambda(\cdot))$ and $\mu(\tau,\lambda(\cdot))$ with the approximations $\tilde{\Pi}(\tau,\lambda(\cdot))$ and $\tilde{\mu}(\tau,\lambda(\cdot))$ presented earlier.

We now turn our attention to a specific example. Suppose $\lambda(t)$ is a step function on each cycle. The simplest example is where $\lambda(t)$ takes on only two values: $\lambda(t)=\lambda_1$ for $t\epsilon[k,k+w_1)$ and $\lambda(t)=\lambda_2$ for $t\epsilon[k+w_1,k+1)$. Here we will set $\lambda_1=9$, $\lambda_2=3$, $\mu=10$ and $w_1=.1$. This could be a crude model for a situation where for a short amount of time we have a very high arrival rate (busy or peak period) while the rest of the day the arrival

rate is moderate. We find from Table 6.1 that a 40x40 matrix should be sufficient for our purpose. Only 5 multiplications of $\underline{P}(1)$ were necessary to reach convergence. To find the mean waiting time, we can use the relationship

$$(6) \qquad \mu(\tau,\lambda(\cdot)) = \frac{E[Q(\tau)]}{\mu}$$

where $Q(\tau)$ is the quasi-limiting value of system size at time τ which was shown by Luchak (1956). Note that $\Pi(\tau,\lambda(\cdot))=P[Q(\tau)=0]$.

We will compare these two quantities, $\Pi(\tau,\lambda(\cdot))$ and $\mu(\tau,\lambda(\cdot))$ with the approximations yielded by the perturbation approach. We need to write $\lambda(t)=\lambda_o(1+\varepsilon\phi(t))$, where $\phi(t)$ is periodic. If we have that

$$\lambda(t) = \begin{cases} \lambda_1 & k\leq t<K+w_1 \\ \lambda_2 & k+w_1\leq t<1 \end{cases} \qquad k=0,1,2,\ldots$$

we can choose λ_o, ε and ϕ_i, i=1,2 such that

$$\phi(t) = \begin{cases} \phi_1 & k\leq t<k+w_1 \\ \phi_2 & k+w_1\leq t<1 \end{cases} \qquad k=0,1,2,\ldots$$

and $\lambda(t) = \lambda_o(1+\varepsilon\phi(t))$.

We would like to be able to write $\phi(t)$ in the form

$$(7) \qquad \phi(t) = \sum_{k=-M}^{M} a_k e^{2\pi ikt}$$

or, in the more traditional form of the Fourier series

$$(8) \qquad \phi(x) = c_o + \sum_{k=1}^{\infty}(c_k\cos 2\pi kx+d_k\sin 2\pi kx)$$

Term by term integration of (8) gives

$$\int_0^1 \phi(x)dx = c_o$$

$$\int_0^1 \phi(x)\cos 2\pi nx dx = \sum_{k=1}^{\infty} c_k \int_0^1 \cos 2\pi kx \cos 2\pi nx dx$$

$$+ \sum_{k=1}^{\infty} d_k \int_0^1 \sin 2\pi kx \cos 2\pi nx dx$$

Now

$$\int_0^1 \cos 2\pi kx \cos 2\pi nx dx = \begin{cases} 0 & k \neq n \\ \frac{1}{2} & k=n \end{cases},$$

and

$$\int_0^1 \cos 2\pi kx \sin 2\pi nx dx = 0.$$

Thus we have that for each n

$$2 \int_0^1 \phi(x)\cos 2\pi nx = c_n.$$

Similarly, we can show

$$2 \int_0^1 \phi(x)\sin 2\pi nx = d_n.$$

For our particular $\phi(t)$,

$$c_n = 2 \int_0^{w_1} \cos 2\pi nx dx + 2 \int_{w_1}^1 \phi_2 \cos 2\pi nx dx$$

$$= \frac{\phi_1}{\pi n} \sin 2\pi n w_1 - \frac{\phi_2}{\pi n} \sin 2\pi n w_1$$

$$= \frac{(\phi_1 - \phi_2)}{\pi n} \sin 2\pi n w_1 \qquad n>0$$

and

$$d_n = 2\int_0^{w_1} \phi_1 \sin 2\pi nx dx + 2\int_{w_1}^{1} \phi_2 \sin 2\pi nx dx$$

$$= -\frac{\phi_1}{\pi n}\cos 2\pi n w_1 + \frac{\phi_1}{\pi n} - \frac{\phi_2}{\pi n} + \frac{\phi_2}{\pi n}\cos 2\pi n w_1$$

$$= \frac{\phi_2 - \phi_1}{\pi n}(\cos 2\pi n w_1 - 1) \qquad n>0$$

$$c_o = \int_0^{w_1} \phi_1 dx + \int_{w_1}^{1} \phi_2 dx$$

$$= (\phi_1 - \phi_2)w_1 + \phi_2.$$

We can now write $\phi(t)$ as in (7) by noting that

$$\phi(x) = c_o + \sum_{k=1}^{\infty}\left[c_k\left(\frac{e^{2\pi ikx} + e^{-2\pi ikx}}{2}\right) + d_k\left(\frac{e^{2\pi ikx} - e^{-2\pi ikx}}{2i}\right)\right]$$

$$= \sum_{k=-\infty}^{\infty} a_k\, e^{2\pi ikx}$$

where

$$a_k = \begin{cases} \frac{c_k}{2} + \frac{d_k}{2i} & k>0 \\ c_o & k=0 \\ \frac{c_k}{2} - \frac{d_k}{2i} & k<0 \end{cases}$$

But our method requires that the series be finite. We can choose an N such that

$$\sum_{k=N+1}^{\infty}(a_k\, e^{2\pi ikx} + a_{-k}\, e^{-2\pi ikx})$$

is small.

To use the perturbation method, we also need $\int_0^1 \phi(x)dx=0$. This requires a transformation of $\lambda(t)$ in the following way. We have now that $\int_0^1 \phi(u)du=a_0=c_0$. Let $\tilde{\phi}(t)=\phi(t)-a_0$ and

$$\begin{aligned}\lambda(t) &= \lambda_0(1+\varepsilon\phi(t)) \\ &= \lambda_0(1+\varepsilon(\tilde{\phi}(t)+a_0)) \\ &= \tilde{\lambda}_0(1+\tilde{\varepsilon}\tilde{\phi}(t))\end{aligned}$$

where

$$\tilde{\lambda}_0 = \lambda_0(1+\varepsilon a_0)$$

$$\tilde{\varepsilon} = \varepsilon/(1+\varepsilon a_0)$$

With the revised values of $\tilde{\lambda}_0$, $\tilde{\varepsilon}$ and $\tilde{\phi}$ we can now use the formulae 4.8 and 4.12 to determine $\tilde{\Pi}(\tau,\lambda(\cdot))$ and $\tilde{\mu}(\tau,\lambda(\cdot))$. We can compute these two quantities graphically with $\Pi(\tau,\lambda(\cdot))$ and $\mu(\tau,\lambda(\cdot))$.

The question arises, though, whether the results will depend on the particular choice of ε, λ_0 and ϕ_i, i=1,2. Let us look at the value of $\tilde{\lambda}_0$. By definition

$$\begin{aligned}\tilde{\lambda}_0 &= \lambda_0(1+\varepsilon a_0) \\ &= \lambda_0(1+\varepsilon(\phi_1-\phi_2)w_1+\varepsilon\phi_2)\end{aligned}$$

Recall that λ_o, ε and ϕ_i were chosen such that $\lambda_1=\lambda_o(1+\varepsilon\phi_o)$, i=1,2, which means also that $\lambda_1-\lambda_2=\varepsilon\lambda_o(\phi_1-\phi_2)$. Thus

$$\tilde{\lambda}_o = \lambda_2+(\lambda_1-\lambda_2)w_1 = \int_o^1 \lambda(u)du.$$

So whatever the choice of parameters, $\tilde{\lambda}_o$ will be the same.

The other terms in $\tilde{\Pi}(\tau,\lambda(\cdot))$ and $\tilde{\mu}(\tau,\lambda(\cdot))$ which involve our parameters are of the form $\tilde{\varepsilon}\tilde{\lambda}_o a_n$, $\tilde{\varepsilon}\tilde{\lambda}_o b_n$ and $\tilde{\varepsilon}\tilde{\lambda}_o\tilde{\phi}(t)$. We have that

$$\tilde{\phi}(t) = \begin{cases} (\tilde{\phi}_1-\tilde{\phi}_2)(1-w_1) & k\le t<w_1+k \\ (\tilde{\phi}_2-\tilde{\phi}_1)W_1 & k+w_1\le t<k+1 \end{cases}$$

$$= \begin{cases} (\phi_1-\phi_2)(1-w_1) & k\le t<w_1+k \\ (\phi_2-\phi_1)w_1 & k+w_1\le t<k+1 \end{cases}$$

and

$$\tilde{\varepsilon}\tilde{\lambda}_o\tilde{\phi}(t) = \frac{\varepsilon}{1+\varepsilon a_o}\lambda_o(1+\varepsilon a_o)\tilde{\phi}(t)$$

$$= \varepsilon\lambda_o\tilde{\phi}(t)$$

$$= \begin{cases} (\lambda_1-\lambda_2)(1-w_1) & k\le t<w_1+k \\ (\lambda_2-\lambda_1)w_1 & k+w_1\le t<k+1 \end{cases}$$

Similarly we can show

$$\tilde{\varepsilon}\tilde{\lambda}_o a_n = \frac{(\lambda_1-\lambda_2)}{\pi n} \sin 2\pi n w_1$$

and

$$\tilde{\varepsilon}\tilde{\lambda}_o b_n = \frac{(\lambda_2-\lambda_1)}{\pi n}(\cos 2\pi n w_1 - 1)$$

Thus we see that our results will be independent of the choice of ε, λ_o, and ϕ_i, i=1,2.

In Figure 6.1 we present the graphs of $\Pi(\tau,\lambda(\cdot))$ and $\tilde{\Pi}(\tau,\lambda(\cdot))$ for the parameters $\lambda_1=9$, $\lambda_2=3$, $\mu=10$. The solid line is the graph produced by the matrix approach, while the triangles form the graph of the approximation. Figure 6.2 displays the two graphs of the mean waiting time. Again, the solid line is the graph produced by the matrix approach, while the triangles form the graph of the approximation. To quantify the comparison we have calculated the average, maximum and minimum values of the vertical distance between the two graphs in each figure. These numbers are based on the values at $\tau=k/50$, k=1,2,...50 and are found in Table 6.1, below.

Table 6.1

	Average	Maximum	Minimum
$\lvert\Pi(\tau,\lambda(\cdot))-\tilde{\Pi}(\tau,\lambda(\cdot))\rvert$	.00317	.02622	.00005
$\lvert\mu(\tau,\lambda(\cdot))-\tilde{\mu}(\tau,\lambda(\cdot))\rvert$	.00106	.00219	.00060

It is clear that in this example, at least, the approximations $\tilde{\Pi}(\tau,\lambda(\cdot))$ and $\tilde{\mu}(\tau,\lambda(\cdot))$ are very good.

FIGURE 6.1

Probability of an Empty Queue

$\lambda_1=9 \quad \lambda_2=3 \quad \mu=10$

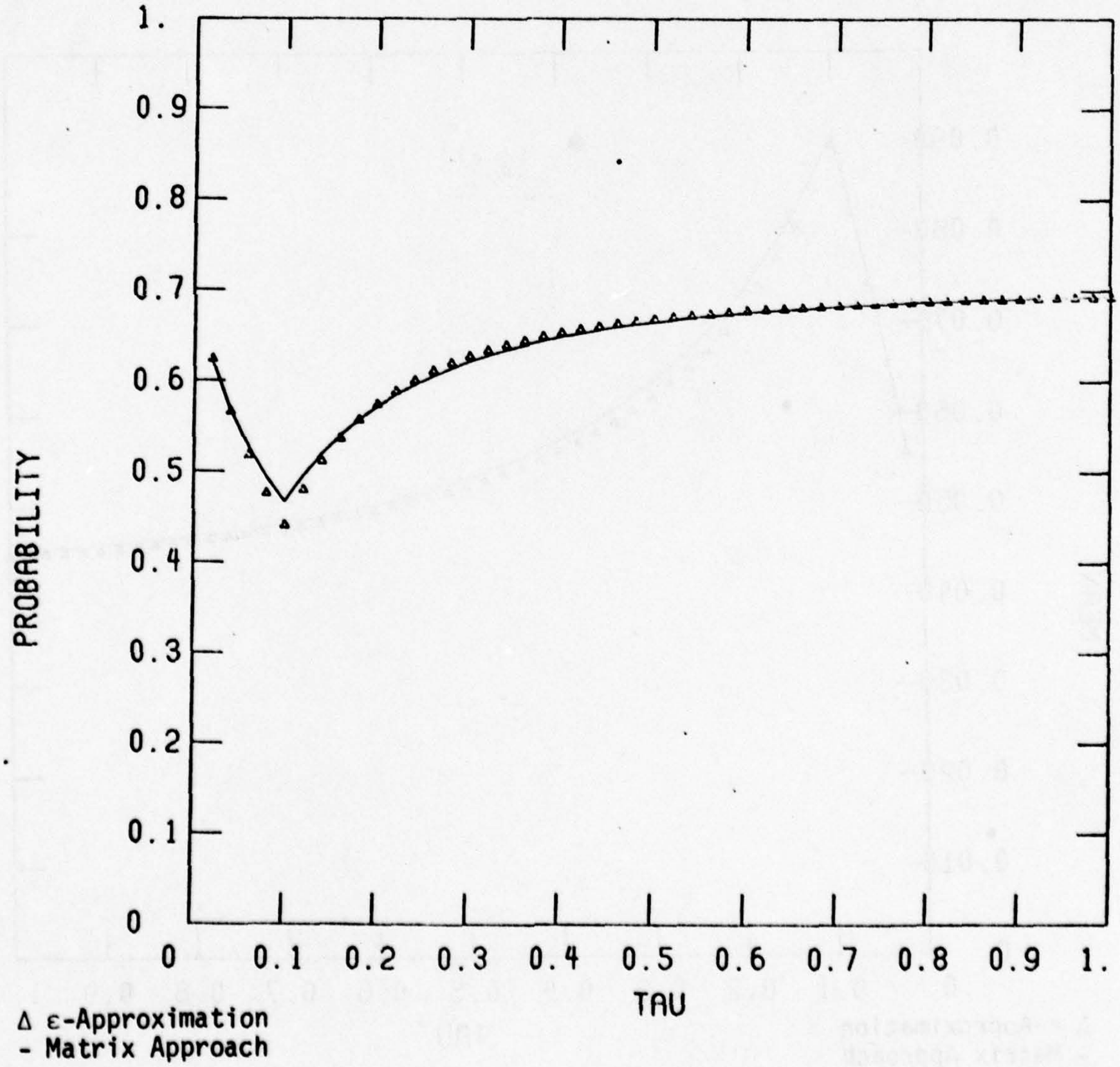
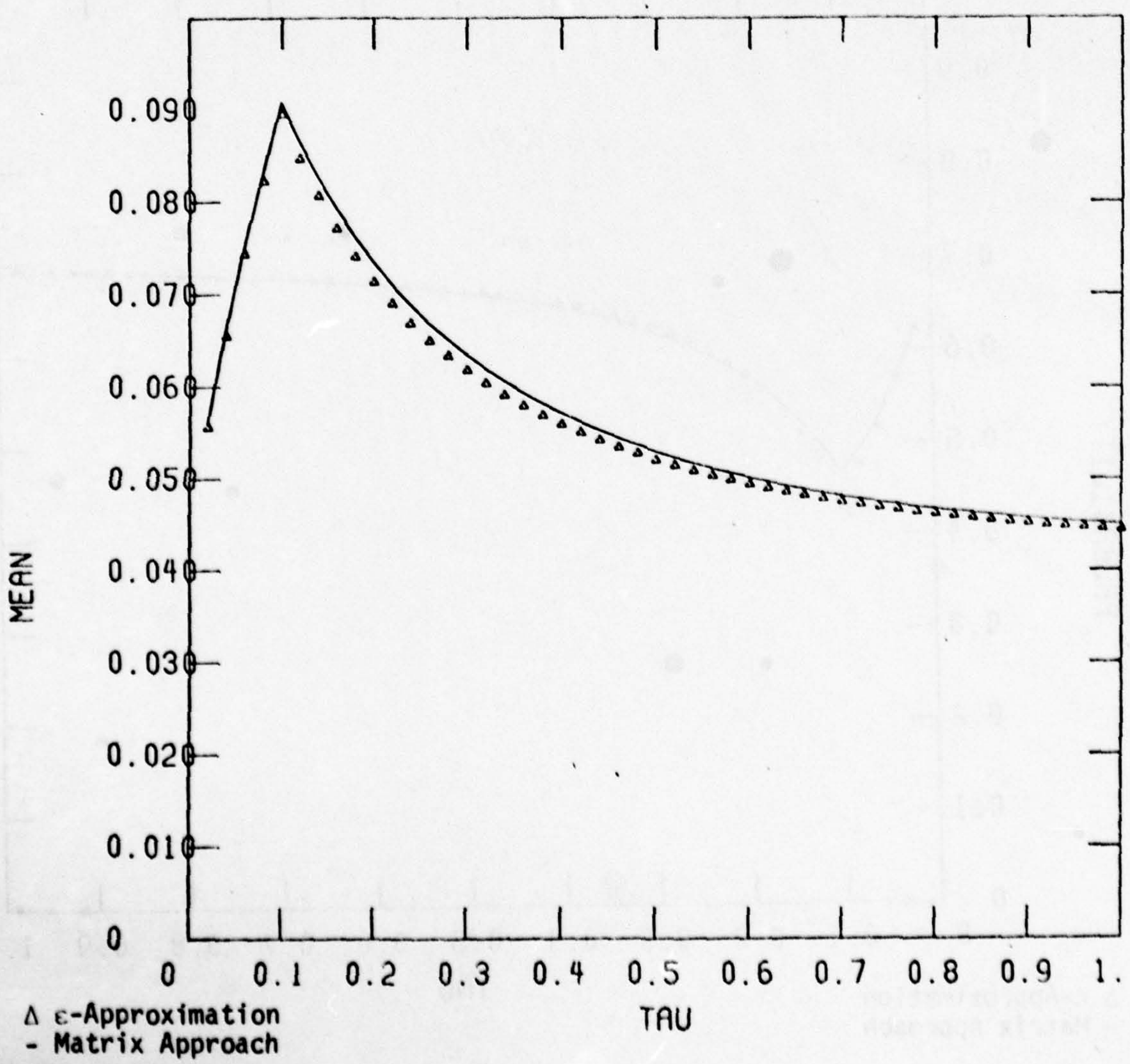


FIGURE 6.2

Mean Waiting Time

$\lambda_1=9 \quad \lambda_2=3 \quad \mu=10$



6.4 Another Example: Periodic $\lambda(t)$

We would like to be able to judge the adequacy of our approximations for situations where $\lambda(t)$ is not a step function. In such cases the matrix approach to the M(t)/M/1 queue can only give an approximation itself. Suppose we were to approximate $\lambda(t)$ above and below by step functions $\lambda^L(t)$ and $\lambda^u(t)$, where $\lambda^L(t) \le \lambda(t) \le \lambda^u(t)$. We know from Lemma 3.2.1 that

$$\Pi(\tau,\lambda^u(\cdot)) \le \Pi(\tau,\lambda(\cdot)) \le (\tau,\lambda^L(\cdot))$$

and

$$\mu(\tau,\lambda^L(\cdot)) \le \mu(\tau,\lambda(\cdot)) \le \mu(\tau,\lambda^u(\cdot)).$$

The matrix approach, at the least, can provide upper and lower bounds to Π and μ. We also know that the closer we approximate the intensity with a step function, the closer that approximation will be to the "true" value. So as we let $\lambda^u(t)-\lambda^L(t)$ become small, $\Pi(\tau,\lambda(\cdot))$ and $\mu(\tau,\lambda(\cdot))$ will be squeezed in between their upper and lower bounds, and we can evaluate the approximations $\tilde{\Pi}(\tau,\lambda(\cdot))$ and $\tilde{\mu}(\tau,\lambda(\cdot))$.

We shall look here at the example in Chapter 4 where $\phi(x)=\cos 2\pi x$. As mentioned earlier, it is computationally efficient to choose equal sized time intervals for the step function. We let $I_k=\left[{}^{(k-1)}/2N, {}^{k}/2N\right)$ where 2N is the number of intervals into which we divide the interval [0,1). Thus we choose for our step functions

$$\lambda^L(t)=\lambda_o(1+\varepsilon\phi_i^L) \quad , \quad t\varepsilon I_i$$

$$\lambda^u(t)=\lambda_o(1+\varepsilon\phi_i^u) \quad , \quad t\varepsilon I_i$$

where

$$\phi_k^u = \begin{cases} \cos(2\pi(k-1)/2n) & k=1, \dots, N \\ \phi_{2N+1-k}^u & k=N+1, \dots, 2N \end{cases}$$

and

$$\phi_k^L = \begin{cases} \cos(2\pi k/2N) & k=1,2, \dots, N \\ \phi_{2N+1-k}^L & k=N+1, \dots, 2N \end{cases}$$

Figures 6.3 and 6.4 display the graphs of the probability of emptiness and mean waiting time, respectively, for $\lambda_o=5$, $\mu=10$, and $\varepsilon=.5$. The solid lines are the graphs of the upper and lower bounds, while the triangles form the graphs of $\tilde{\Pi}(\tau,\lambda(\cdot))$ and $\tilde{\mu}(\tau,\lambda(\cdot))$. We see from the figures that, at least for the approximations, $\tilde{\Pi}$ and $\tilde{\mu}$ are quite good. Here we have used 80 intervals for the step function. Since the average difference between the upper and lower bound is .0125 for Π and .005 for μ, this seems to be a sufficiently fine step function approximation.

We can make a few comparisons of interest. First we look at the deviation between the perturbation approximations and their bounds. It is also interesting to consider the mean waiting time and probability of emptiness of a queue which has the homogeneous intensity λ_o- the average intensity over a period. This is the queue that is generally used as a substitute for the periodic queue. This can give us a basis of comparison to judge the error our approximation gives us. In the following two tables we present the average, maximum and minimum absolute differences between the bounds of $\Pi(\tau,\lambda(\cdot))$ and $\mu(\tau,\lambda(\cdot))$ and the values obtained from the two queueing models. We let

$$D_1 = |\psi(\tau,\lambda^L(\cdot))-\psi(\tau,\lambda_o)|$$
$$D_2 = |\psi(\tau,\lambda^L(\cdot))-\tilde{\psi}(\tau,\lambda(\cdot))|$$
$$D_3 = |\psi(\tau,\lambda^u(\cdot))-\psi(\tau,\lambda_o)|$$
$$D_4 = |\psi(\tau,\lambda^u(\cdot))-\tilde{\psi}(\tau,\lambda(\cdot))|.$$

where $\psi=\Pi$ or μ, as appropriate

Table 6.2 Probability of Emptiness

	D_1	D_2	D_3	D_4
Average	.07334	.00687	.07219	.00625
Minimum	.00076	.00025	.00286	.00064
Maximum	.12728	.01415	.11394	.01243

Table 6.3 Mean Waiting Time

	D_1	D_2	D_3	D_4
Average	.02013	.00093	.01966	.00596
Minimum	.00052	.00033	.00012	.00509
Maximum	.03722	.00153	.03229	.00690

Since $D_1>D_2$ and $D_3>D_4$ it seems apparent that the approximations for the periodic queue are better to use than the results for the homogeneous queue. It also seems that $\tilde{\Pi}(\tau,\lambda(\cdot))$ and $\tilde{\mu}(\tau,\lambda(\cdot))$ are sufficiently good approximations that they should be the choice over the matrix results, particularly when ease of computation is considered.

FIGURE 6.3

Probability of an Empty Queue

λ_o=5.0 μ=10.0

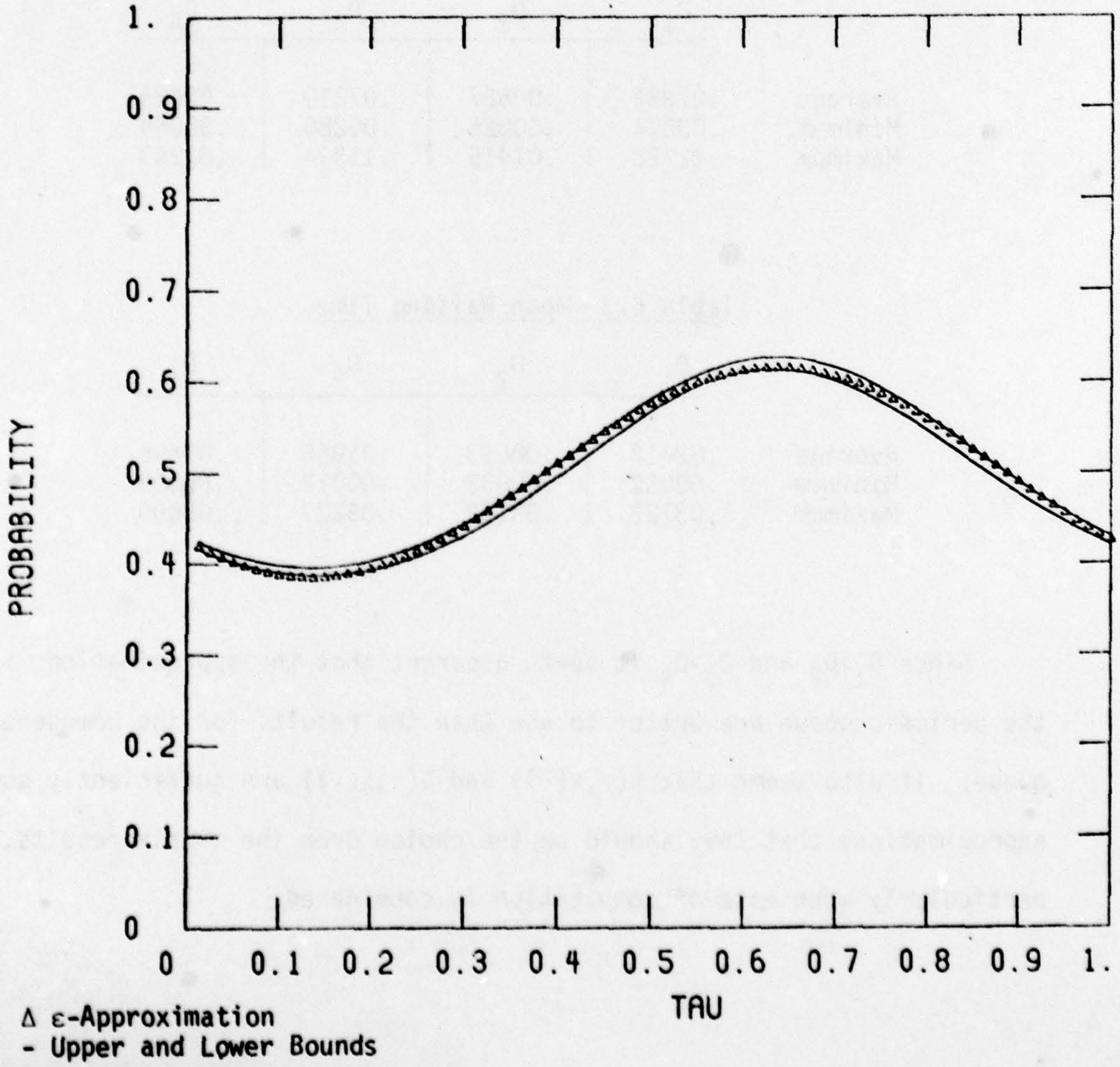
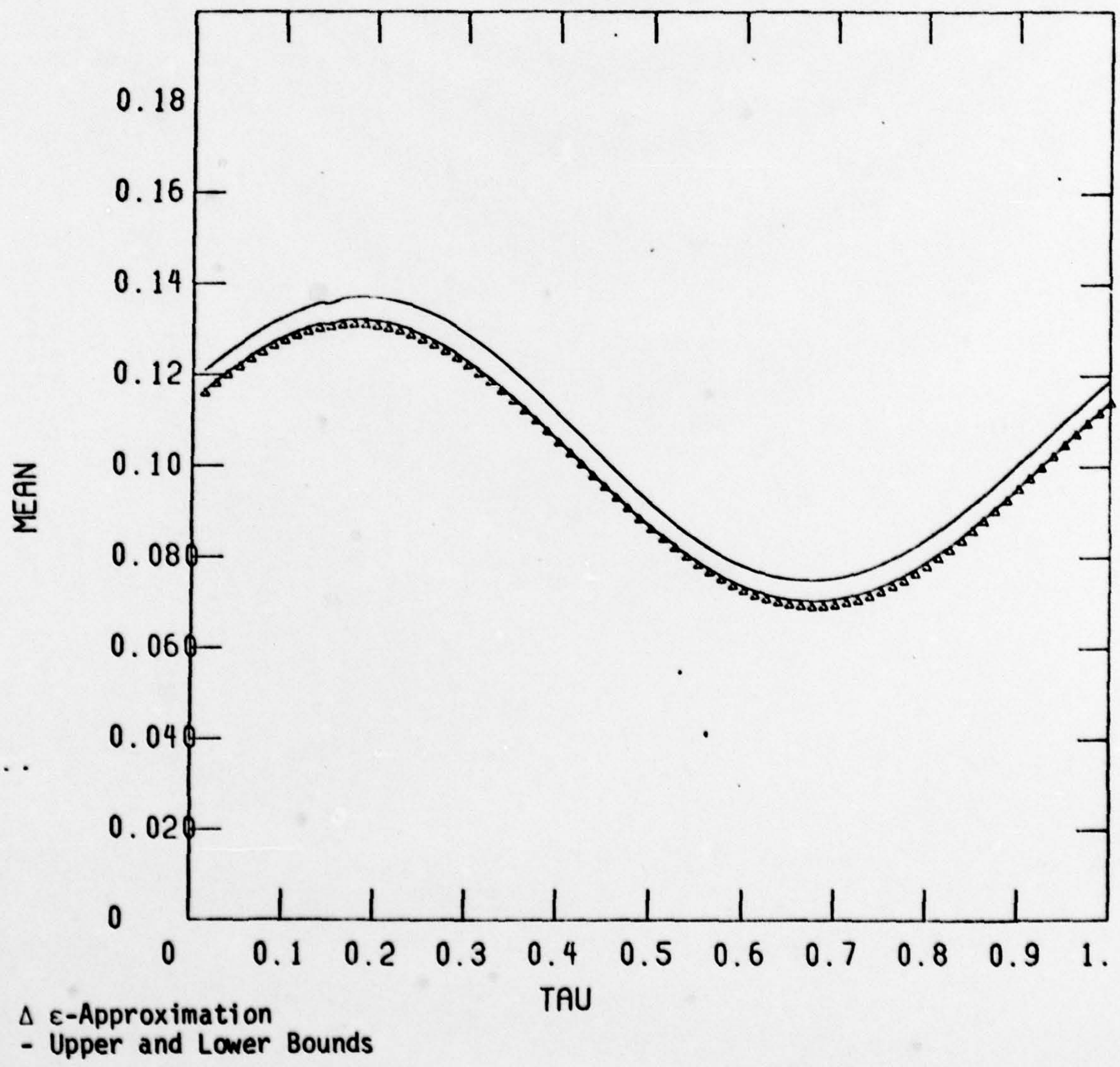


FIGURE 6.4

Mean Waiting Time

$\lambda_o=5.0 \quad \mu=10.0$



Δ ε-Approximation
- Upper and Lower Bounds

CHAPTER 7

STORAGE SYSTEMS WITH STOCHASTIC INPUT INTENSITY

7.1 The Method

The input process of a storage system will not always be a deterministic function of time. It might seem more realistic to assume that the rate of input varies randomly over time. This would be the case, for example, if inputs to the system originate from a variety of different sources and the number of sources is a random variable. We would like to be able to examine processes such as these in which the input rate is generated by some stochastic mechanism. In particular, it would be good if the method employed previously on storage processes with nonhomegeneous Compound Poisson input could be adapted to allow for stochastic $\lambda(t)$. Fortunately, it is a relatively simple matter to do this.

As before, we model our arrival intensity as $\lambda(t)=\lambda_0(1+\varepsilon\phi(t))$. We assume now, though, that $\phi_\omega(t)=\phi(t,\omega)$ is a measurable stochastic process such that $E\{\phi(t,\omega)\}$ exists and $\int_0^t E|\phi(x,\omega)|dx<\infty$, $t>0$. If we have a realization of the process, $\phi_\omega(t)$, then $\lambda(t)=\lambda_0(1+\varepsilon\phi\omega(t))$ is just as before in the deterministic case. Thus the probability of an empty store, given that realization, is just the conditional probability

$$\Pi_\phi(t,\lambda(\cdot),W_0) = P\{z(t)=0|\phi_\omega(s) \quad 0\leq s\leq t\}$$

$$(1) = \Pi(t,\lambda_0,W_0)+\varepsilon\lambda_0 \int_0^t \phi_\omega(u)\hat{\Pi}(t-u,\lambda_0,W_1)du+O(\varepsilon^2).$$

As long as $E\phi_\omega(u)$ exists and $\int_0^t E|\phi_\omega(u)|du<\infty$, the integral in (1) will exist. We can make this claim, since $|\hat{\Pi}\phi|<|\phi|$ implies that $E\{\phi_\omega(u)\hat{\Pi}(t-u,\lambda_0,W_1)\}$ exists and $\int_0^t E|\phi_\omega(u)\hat{\Pi}(t-u,\lambda_0,W_1)|du<\infty$. We note that Π_ϕ is itself now a random variable.

To obtain the unconditional probability we can take the expectation of Π_ϕ with respect to the distribution of ϕ_ω. Then

$$\begin{aligned}\Pi(t,\lambda(\cdot),W_0)&=EP\{Z(t)=0|\phi_\omega(s),\ 0\leq s\leq t\}\\ &=\Pi(t,\lambda_0,W_0)+\varepsilon\lambda_0 E\left[\int_0^t \phi_\omega(u)\hat{\Pi}(t-u,\lambda_0,W_1)du\right]+O(\varepsilon^2)\end{aligned}$$

By Fubinis Theorem we can move the expectation inside the integral to get

$$(2)\quad \Pi(t,\lambda(\cdot),W_0)=\Pi(t,\lambda_0,W_0)+\varepsilon\lambda_0\int_0^t m(u)\hat{\Pi}(t-u,\lambda_0,W_1)du+O(\varepsilon^2)$$

where $m(\cdot)$ is the mean function.

We can also find the mean content of the storage process by using the relationship $EZ(t)=E\left[E(Z(t)|\phi_\omega)\right]$. If $E\{\phi_\omega(u)\hat{\mu}(t-u,\lambda_0,W_1)\}$ exists and $\int_0^t E\left[|\phi_\omega(u)\hat{\mu}(t-u,\lambda_0,W_1)|\right]du<\infty$ then

$$(3)\quad \mu(t,\lambda(\cdot),W_0)=\mu(t,\lambda_0,W_0)+\varepsilon\lambda_0\int_0^t m(u)\hat{\mu}(t-u,\lambda_0,W_1)du+O(\varepsilon^2)$$

It appears that the complex behavior of the stochastic input process is captured in the mean of the arrival rate. If $m(t)\equiv 0$ then $\Pi(t,\lambda(\cdot),W_0)=\Pi(t,\lambda_0,W_0)+O(\varepsilon^2)$ and $\mu(t,\lambda(\cdot),W_0)=\mu(t,\lambda_0,W_0)+O(\varepsilon^2)$. Unfortunately, in most situations of interest this will not be the case. We note that the formulae in (2) and (3) are now the same as those in Chapter

3, with m(.) in place of $\phi(.)$. Thus to look at the limiting distribution of Π and μ we are now in the same situation as that Chapter.

It is unlikely that the mean function will be periodic, so much of the results we have looked at will not apply. A more likely event is that $m(t)=a+b(t)$ where $b(t)\varepsilon L_1$. Then

$$\Pi(t,\lambda(.),W_0)=\Pi(t,\lambda_0,W_0)+\varepsilon\lambda_0 a\int_0^t \hat{\Pi}(t-u,\lambda_0,W_1)du$$

$$+\ \varepsilon\lambda_0 \int_0^t b(u)\hat{\Pi}(t-u,\lambda_0,W_1)du+O(\varepsilon^2).$$

Since $\hat{\Pi}$ is a bounded function which converges to a limit (which is 0) we can use Lemma 3.2.2 to show that

$$\lim_{t\to\infty} \int_0^t b(u)\hat{\Pi}(t-u,\lambda_0,W_1)du=\hat{\Pi}(\infty,\lambda_0)\int_0^\infty b(u)du=0$$

This leads to the limit

(4) $$\lim_{t\to\infty} \Pi(t,\lambda(.),W_0)=\Pi(\infty,\lambda_0)+\varepsilon\lambda_0 a\int_0^\infty \hat{\Pi}(u,\lambda_0,W_1)du+O(\varepsilon^2)$$

7.2 An Example: A Queue with Two Phases

The most commonly studied process with stochastic arrival rates is the queue in which the arrival rate depends on the "phase" of the queue at the time of arrival, and the amount of time spent in a phase is a negative exponential random variable. This essentially yields an intensity $\lambda(.)$ which is a step function which changes levels at random times. This queueing process has been discussed by Yechiali and Naor, Eisen and Tainiter and Neuts, to name a few. While the simple queueing problem

with only two phases has been studied extensively with traditional methods, it is informative to look at this example in the light of the alternative method presented here.

Suppose we have a situation where the arrival rate fluctuates between two levels $\lambda_1=\lambda_o(1+\epsilon\phi_1)$ and $\lambda_2=\lambda_o(1+\epsilon\phi_2)$, where the time spent at level i has a negative exponential distribution with mean $1/\eta_i$. To find Π and μ we need to determine the mean of ϕ. We can write m(t) as $m(t)=\phi_1P_1(t)+\phi_2P_2(t)$ where

$$P_i(t)=P\{\text{system is in phase } i \text{ at time } t\}$$

The phase process forms a simple two state Markov Chain. We can use the Kolmogorov forward equations to determine $P_i(t)$, i=1,2, as follows.

$$P_1(t+dt)=P_1(t)(1-\eta_1dt)+P_2(t)\eta_2dt+o(dt)$$

(5) $$P_1'(t) = -\eta_1P_1(t)+\eta_2P_2(t)$$

Since $P_1(t)+P_2(t)=1$ (5) is

$$P_1'(t) = \eta_2-(\eta_1+\eta_2)P_1(t)$$

Solving this differential equation we get

$$P_1(t) = \frac{\eta_2}{\eta_1+\eta_2} - \left(\frac{\eta_2}{\eta_1+\eta_2} - P_1(0)\right)e^{-(\eta_1+\eta_2)t}$$

(6)

$$P_2(t) = \frac{\eta_1}{\eta_1+\eta_2} + \left(\frac{\eta_2}{\eta_1+\eta_2} - P_1(0)\right)e^{-(\eta_1+\eta_2)t}$$

We can now write m(t) as

$$m(t) = \phi_2+(\phi_1-\phi_2)P_1(t)$$

$$= \frac{\phi_2+(\phi_1-\phi_2)\eta_2}{\eta_1+\eta_2}-(\phi_1-\phi_2)\left(\frac{\eta_2}{\eta_1+\eta_2}-P_1(0)\right)e^{-(\eta_1+\eta_2)t}$$

If we let $a=\phi_2+(\phi_1-\phi_2)\eta_2/(\eta_1+\eta_2)$ and

$b(t)=-(\phi_1-\phi_2)\left[\eta_2/(\eta_1+\eta_2)-P_1(0)\right]e^{-(\eta_1+\eta_2)t}$ then we see from (4) that

$$\text{(7)}\quad \lim_{t\to\infty}\Pi(t,\lambda(.),W_o)=\Pi(\infty,\lambda_o)+\varepsilon\lambda_o\left[\frac{\phi_2+(\phi_1-\phi_2)\eta_2}{\eta_1+\eta_2}\right]\int_0^\infty\hat{\Pi}(u,\lambda_o,W_1)du+O(\varepsilon^2)$$

We recall from Chapter 4 that for an M/M/1 queue, if $W_o=\tilde{W}$ then

$$\int_0^\infty\hat{\Pi}(u,\lambda_o,W_1)du = -\gamma_1\Pi(\infty,\lambda_o)$$

$$= -\frac{(1-\lambda_o/\mu)}{\mu-\lambda_o}$$

$$= -\frac{1}{\mu}$$

Substituting into (7) we have

$$\text{(8)}\quad \lim_{t\to\infty}\Pi(t,\lambda(.),\tilde{W}) = 1-\frac{\lambda_o}{\mu}-\frac{\varepsilon\lambda_o}{\mu}\left[\frac{\phi_2+(\phi_1-\phi_2)\eta_2}{\eta_1+\eta_2}\right]+O(\varepsilon^2)$$

We can use the relationship between the parameters λ_i, ϕ_i and ε to rewrite this as

(9) $\lim_{t\to\infty}\Pi(t,\lambda(.),\tilde{W}) = 1 - \frac{\lambda 2}{\mu} + \frac{(\lambda_2-\lambda_1)\eta_2}{\mu(\eta_1+\eta_2)} + O(\varepsilon^2)$

We notice that, as we would expect, the limit does not depend on the choice of ε, λ_o, ϕ_i.

It is interesting to compare the approximation (9) with the actual value as found in the literature. Yechiali and Naor (1971) present the relationship

(10) $$p_{1o}\mu_1 + p_{2o}\mu_2 = \hat{\mu}-\hat{\lambda}$$

where p_{io}=P{Z(t)=0 and system is in phase i}, i=1,2 in steady state, $\hat{\lambda}=p_{1.}\lambda_1+p_{2.}\lambda_2$, $\hat{\mu}=p_{1.}\mu_1+p_{2.}\mu_2$ and $p_{i.}=\lim_{t\to\infty} P_i(t)$. If $\mu_1=\mu_2=\mu$ then (10) simplifies to

(11) $$p_{1o} + p_{2o} = 1-\hat{\lambda}/\mu.$$

But $p_{1o}+p_{2o}$=P{Z(t)=0}. Thus we have that

$$\lim_{t\to\infty} \Pi(t,\lambda_o,W) = 1 - \hat{\lambda}/\mu.$$

Notice that from equation (6) we have that

(12) $$p_{1.} = \eta_2/(\eta_1+\eta_2) \text{ and } p_2 = \eta_1/(\eta_1+\eta_2)$$

If we substitute (12) into (9) and rearrange some terms, we find that $O(\varepsilon^2)=0$.

Yechiali and Naor (1971) state that $p_{1o}+p_{2o}=1-\hat{\lambda}/\hat{\mu}$ if and only if $\lambda_1/\mu_1=\lambda_2/\mu_2$. While it is true that $\lambda_1/\mu_1=\lambda_2/\mu_2$ implies that

$p_{1o}+p_{2o}=1-\hat{\lambda}/\hat{\mu}$, we have already shown in (11) that this is not the only circumstance under which this relationship holds. Their theorem does indeed hold true when $\mu_1 \neq \mu_2$. But when $\mu_1=\mu_2$ the proof fails because at one point they inadvertently divide by zero.

APPENDIX

A REVIEW OF STANDARD RESULTS

The standard storage theory model is the following: Inputs to the system occur at points t_0, t_1, . . . where the time between inputs $X_i=t_i-t_{i-1}$ are i.i.d random variables. The size of each input, U_i, is governed by a distribution B(x) with B(0+)=0. Thus input to the store can be described by the sequence $\{X_1, U_1, X_2, U_2, \ldots\}$. In many instances the X_i are assumed to be from a negative exponential distribution. In such cases the total input into the system is Compound Poisson.

The content of the store, in standard theory, is released at a constant rate c (r(x)=c) as long as the store is not empty. Thus in an interval of length y, the output of the store will be at most c y, with the maximum being achieved if the store has positive content throughout the interval. When the rate of release is 1, the storage process is equivalent to a single server queue. In this case the X_i are interpreted as interarrival times and B(x) is the service time distribution. A simple change of time scale will convert a storage process with r(x)=c into one with r(x)=1. Thus without loss of generality we can focus our attention on the single server queue, where much of the literature is found.

We cite here results of the M/G/1 queue which concern us in this research. We look to the dissertation of Smith (1953) to summarize the results for this appendix.

The following notation will be needed:

$G(x)$ is the distribution function of the busy period

$W(x,t)$ is the distribution function of the waiting time at t

β_j is the j^{th} moment of service time

$\mu(t)$ is the mean of the waiting time at t

λ is the arrival intensity.

This notation agrees with the notation used in later chapters and is different from that used originally in Smith (1953).

The first theorem and its corollary provide information about the event ε, which occurs at t if $Z(t)=0$ and $Z(t-)>0$, through an examination of the busy period.

Theorem 28: If $\lambda\beta_1<1$, $G(x)$ possesses as many moments as $B(x)$, and if the j^{th} moment of $B(x)$ is β_j the moments γ_j of $G(x)$ are given by

$$\gamma_1 = \beta_1/(1-\lambda\beta_1)$$

$$\gamma_2 = \beta_2/(1-\lambda\beta_1)^3$$

$$\gamma_3 = \{\beta_2+3\lambda\beta_2^2/(1-\lambda\beta_1)\}/(1-\lambda\beta_1)^4$$

Corollary 28.1: Assuming $B(+\infty)=1$ we have that

(i) If $\lambda\beta_1\leq 1$, $G(+\infty)=1$

(ii) If $\lambda\beta_1>1$, $G(+\infty)=B^*(\sigma_0)$

where σ_0 is the root in $R(s)>0$ of the equation $s-\lambda(1-B^*(s))=0$.

We see from these theorems that the condition $\lambda\beta_1\leq 1$ is sufficient to ensure that the event ε is certain. The next theorem demonstrates

conditions under which a limiting distribution of waiting time exists.

Theorem 30: (i) If $\lambda\beta_1<1$ and $W(\infty,0)=1$ then there is proper distribution function $W(x)$ such that

$$\lim_{t\to\infty} W(x,t) = W(x)$$

where

$$W^*(s) = \int_0^\infty e^{-sx}\, dW(x) = \frac{s(1-\lambda\beta_1)}{s-\lambda(1-B^*(s))}$$

(ii) If $\lambda\beta_1\geq 1$ then for all $x\geq 0$

$$\lim_{t\to\infty} W(x,t) = 0$$

We can gain some additional information about the queue which we will need from $W^*(s)$. First, if we take the limit of $W^*(s)$ as s goes to infinity we find that for the queue in steady state, the probability of an empty queue is $(1-\lambda\beta_1)$. Second, if we look at

$\lim_{s\to\infty} \frac{-dW^*(s)}{ds}$ we find that the mean waiting time in steady state is $\lambda\beta_2/2(1-\lambda\beta_1)$. These are two well known and often used facts.

The last theorem we present is an estimate of $I(t)$, the expected idle time in $(0,t)$ and $\mu(t)$.

Theorem 32: If $\lambda\beta_1<1$, $\beta_2<\infty$, $\mu(0)<\infty$ then as $t\to\infty$

$$I(t) = (1-\lambda\beta_1)t + \frac{\lambda\beta_2}{2(1-\lambda\beta_1)} - \mu(0) + o(1)$$

and

$$\mu(t) = \frac{\lambda\beta_2}{2(1-\lambda\beta_1)} + o(1).$$

BIBLIOGRAPHY

Brockwell, P. J. (1977). "Stationary Distributions for Dams with Additive Input and Content-Dependent Release Rate", Advances in Applied Probability. 9:645-663.

Cinlar, E. and Pinsky, M. (1971). "A Stochastic Integral in Storage Theory", Zeitschrift fur Wahrscheinlichkeitstheorie und Verwandte Gebiete. 25:227-240.

Clarke, A. B. (1956). "A Waiting Time Process of Markov Type", Annals of Mathematical Statistics. 27:452-459.

Eisen, M. and Tainiter, M. (1963). "Stochastic Variations in Queueing Processes", Operations Research. 11:922-927.

Feller, W. (1968). An Introduction to Probability Theory and Its Applications. 3rd. ed. New York: John Wiley and Sons, Inc.

Gani, J. (1962). "A Stochastic Dam Process with Nonhomogeneous Poisson Input", Studio Mathematica. 11:307-315.

Harrison, J. M. and Lemoine, A. J. (1977). "Limit Theorems for Periodic Queues", Journal of Applied Probability. 14:566-576.

Harrison, J. M. and Resnick, S. J. (1976). "The Stationary Distribution and First Exit Probabilities of a Storage Process with General Release Rule", Mathematics of Operations Research. 4:347-358.

Hasofer, A. M. (1964). "On the Single Server Queue with Nonhomogeneous Poisson Input and General Service Time", Journal of Applied Probability. 1:369-384.

________. (1965). "Some Perturbation Results for the Single Server Queue with Poisson Input", Journal of Applied Probability, 2:462-466.

Koopman, B. O. (1972). "Air Terminal Queues under Time Dependent Conditions", Operations Research. 20:1089-1114.

Leese, E. L. (1965). "Numerical Methods of Determining the Transient Behavior of Queues with Variable Arrival Rate", Queueing Theory: Recent Developments and Applications. Ed. R. Cruon. New York: American Elsevier Publishing Company, pp. 86-97.

Luchak, G. (1956). "The Solution of the Single Channel Queueing Equation Characterized by a Time Dependent Poisson Distributed Arrival Rate and a General Class of Holding Times", Operations Research. 4:711-732.

Moore, S. C. (1975). "Approximating to the Behavior of Non-Stationary Single Server Queues", Operations Research. 23:1011-1033.

Moran, P. A. P. (1967). "Dams in Series with Continuous Release", Journal of Applied Probability. 4:380-388.

________. (1969). "A Theory of Dams with Continuous Input and a General Release Rule", Journal of Applied Probability. 6:88-98.

Neuts, M. F. (1971). "Queues Subject to Extraneous Phase Changes", Advances in Applied Probability. 3:78-119.

Reich, E. (1958). "On the Integro-differential Equation of Takács, I. Annals of Mathmatical Statistics. 29:563-570.

________. (1959). "On the Integro-differential Equation of Takács, II. Annals of Mathematical Statistics. 30:143-148.

Sakr, L. (1971). "Asymptotische Eigenshaften von Wartesystemin mit Periodischer Eingangsintensitat", Mathematische Operationforschung und Statistik. 2:227-236.

Smith, W. L. (1953). Stochastic Sequences of Events. Ph.D. Dissertation, Pembroke College.

________. (1954). "Asymptotic Renewal Theorems", Proceedings of the Royal Society of Edinburgh. 64:9-48.

________. (1955). "Regenerative Stochastic Processes", Proceedings of the Royal Society of London, Series A. 232:6-31.

Takács, L. (1955). "Investigation of Waiting Time Problems by Reduction to Markov Processes", Acta Mathematica Budapest. 6:101-129.

Wragg, A. (1963). "The Solution of an Infinite Set of Differential-Difference Equations Occurring in Polymerization and Queueing Problems", Proceedings of the Cambridge Philosophical Society. 59:117-124.

Yechiali, U. (1973). "A Queueing Type Birth-Death Process Defined on a Continuous Time Markov Chain", Operations Research. 21:604-609.

Yechiali, U. and Naor, P. (1971). "Queueing Problems with Heterogeneous Arrivals and Services", Operations Research. 17:722-734.

Yevdokimova, G. S. (1974). "The Distribution of Waiting Time in the Case of a Periodic Input Flow", Engineering Cybernetics. 12:81-85.